LETTER FROM THE EDITOR

I am writing this in late June. The last several weeks have seen mass protests for the causes of social justice and racial equality, and this has forced people in all areas of American life to open their eyes to the realities of systemic racism. The leadership of the MAA has released two statements in that regard: a joint statement from President Michael Dorff and Executive Director Michael Pearson [1], and a separate statement from Carrie Diaz Eaton, who chairs the Committee on Minority Participation in Mathematics [2]. I would encourage everyone reading this to have a look at their statements.

Mathematics Magazine exists to publish high-quality, expository mathematical writing. However, we take a very expansive view of what constitutes "mathematical writing." In my February 2020 letter introducing myself as the new editor, I wrote, "Mathematics is more than theorems, equations, and ever more complex variants on well-known problems. I believe that something called *Mathematics Magazine* should cover mathematics in all its forms, and not every article needs to be a notation-filled, jargon-fest. . . . When the opportunity arises, I want to publish articles that are going to inspire conversation."

In the light of recent events, I feel these sentiments more strongly than ever. There is a culture to mathematics, and cogent analyses and discussions of that culture are appropriate topics for this magazine just as surely as anything from calculus or geometry. I am very interested in reading submissions that have a clear voice, especially if they open people's eyes to often marginalized points of view. If *Mathematics Magazine* can provide a platform for good writing in this vein, then I will consider my editorship to have been a success.

We have the usual bumper crop of articles for you in this issue, starting with an introduction to KAM theory by James Walsh. "KAM" is an acronym for "Kolmogorov–Arnol'd–Moser," and KAM theory addresses certain physical problems about oscillating systems. Mathematically it is a beautiful synthesis of differential equations and number theory. It is very heady stuff, but Walsh's introduction is both lucid and engaging.

The graph theorists among you will find plenty of food for thought. Eric McDowell solves a problem in the theory of Cayley digraphs. Cayley introduced the graphs that now bear his name as a visual device for studying groups. But given a specific directed graph, how can we tell if there is a group out there for which it is a Cayley digraph? From there you can move on to Warut Suksumpong's article about graph cycles and olympiad problems. He shows that several ingenious problems from international mathematics competitions, none of which seem to have anything to do with graph theory, can all be solved by applying a few elementary results about cycles in graphs. Suksompong's solutions are exceedingly clever, and I think you will enjoy working through them as much as I did.

Number theory enthusiasts will enjoy sinking their teeth into the article by Zafer Selcuk Aygin and Kenneth Williams. They consider prime factorizations of Fermat numbers, meaning numbers of the form $2^k + 1$. Fermat famously conjectured that if k is a power of two, then $2^k + 1$ is prime, but this was shown to be false when Euler managed to factor $2^{32} + 1$. The state of the art has advanced since then, and Aygin and Williams have many insightful things to say about it.

We also have two articles taking sophisticated looks at more elementary topics. Marc Chamberland, Shida Jing, and Sanah Suri discuss the one-seventh ellipse. I must confess that this one was new to me, but it is one of those astonishing results that makes you say, “Gosh! *Somebody* noticed that.” The decimal expansion of $1/7$ is 0.142857 repeating. If you treat consecutive pairs of digits as points in the plane you get $(1, 4)$, $(4, 2)$, $(2, 8)$, $(8, 5)$, $(5, 7)$ and $(7, 1)$, and these points all lie on an ellipse. It still works if you use two-digit numbers: the points $(14, 28)$, $(42, 85)$, $(28, 57)$, $(85, 71)$, $(57, 14)$, and $(71, 42)$ also lie on an ellipse. Amazing! For his part, Roger Nelsen takes a fresh look at well-known trigonometric identities attributed to Huygens, Cusa, and Wilker. He provides fresh proofs using little more than basic calculus and a lot of ingenuity.

From David Bressoud, a former President of the MAA, we have a stirring essay describing an unconventional approach to teaching real analysis. Allison Henrich interviews Henry Segerman, who employs some very innovative approaches to mathematical art and visualization. We round out the issue with Proofs Without Words, Problems, and Reviews.

Jason Rosenhouse, Editor

REFERENCES

[1] Dorff, M., Pearson, M. (2020). A message from MAA leadership: Mathematics, and MAA, for human flourshing. mathvalues.org/masterblog/message-from-maa-leadership

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A Window into the World of KAM Theory

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What happens if you periodically force a system that is already oscillating? The reader almost surely will have had experience investigating this question—in days gone by perhaps—when on a swing at the local playground. (When should I "kick" for optimal height?) Many readers of this MAGAZINE will have taught or been exposed to this question in the sophomore-level differential equations course, where a periodically forced mass-on-a-spring mechanical model is a familiar topic. In the absence of damping, important behaviors such as beats and resonance can occur. Typically, however, a *linear* spring is considered in this model. That is, the force exerted by the spring varies linearly with the displacement of the mass. This places severe restrictions on the types of behavior the model can exhibit. In particular, periodically forced linear oscillators do not behave *chaotically*. Nearly identical initial conditions do not lead to diverging trajectories, a property of chaotic dynamical systems known as *sensitive dependence on initial conditions*. Indeed, the difference between two solutions for the undamped, linear mass-spring model varies periodically, a fact any reader having had a course in ordinary differential equations (ODEs) can deduce.

A periodically forced *nonlinear* oscillator can exhibit much more interesting behavior! One classic example is provided by the periodically forced pendulum, either with damping [**16**] or without [**15**]. In the former case the exact interplay between gravity, damping, and forcing responsible for chaotic behavior can be difficult to discern. This led a former student of mine, Jaie Woodard, to create a simpler periodically forced and damped oscillator model that she numerically analyzed via *Mathematica*. This work comprised Jaie's (wonderful) project for a junior-level dynamical systems course. All of the rich behavior occurring in the forced damped pendulum occurs in Woodard's simpler model, including the period-doubling route to chaos, a symbolic dynamics interpretation of model behavior, and fractal basin boundaries [**30**].

This article can be viewed, in part, as a companion piece to the paper of Woodard and Walsh [**30**]. Here, Woodard's model will be periodically forced in the absence of damping. While chaotic behavior can again be realized, the theory behind the analysis is much different in this case. When the model is damped, volumes in phase space are reduced over time as system solutions evolve. In the absence of damping, however, phase space volumes are preserved, akin to incompressible fluid flow. The property of volume preservation occurs in all systems for which there is a conserved quantity such as total energy, which will be shown is the case for Woodard's model. It is in the setting of volume preserving systems that Kolmogorov–Arnol'd–Moser (KAM) theory plays its essential role.

My overarching hope is to provide insight, in the simplest of settings, into the power of KAM theory. With the beautiful interplay of differential equations (including chaos

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and fractal theory) with deep results from number theory, KAM theory is one of the profound mathematical achievements of the last century. In this article I will indicate the nature of these deep connections, thereby shedding light on the essence of these extraordinary results. I refer the reader to Dumas [11] for a captivating account of the history of KAM theory, and to sources such as De la Llave [10], Dumas [11], and Wayne [29] for a deeper look at the mathematics involved.

Let's begin with an abridged history of KAM theory.

A (very) brief history

Named after contributions from A. Kolmogorov, V. Arnol'd, and J. Moser, KAM theory concerns perturbations of *Hamiltonian systems*, as discussed in the context of Woodard's model in the following sections. For a bit of intuition, imagine a nonlinear planar system of ODEs corresponding to a mechanical model in which energy is conserved. The system trajectories would exhibit complete regularity in the sense that each would lie on a level set of a (Hamiltonian) function $H : \mathbb{R}^2 \to \mathbb{R}$ representing total system energy (as in Figure 2). What happens to this structure if a small perturbation, such as a small amplitude periodic forcing, is applied? Does the regular, stable behavior evident in Figure 2 persist, or might more complicated behavior arise? In the first half of the 20th century, many believed it was one or the other: either the perturbed system is again completely regular (*completely integrable* is the technical term), or regularity is lost, with trajectories wandering throughout regions in the plane while never settling down. KAM theory says that it is a little of both, a mixture of persistent structure and chaotic behavior.

The first major contribution to KAM theory was provided by Kolmogorov in a 1954 address at the International Congress of Mathematicians in Amsterdam. Kolmogorov stated—and outlined a proof of—a "KAM theorem" in the class of analytic Hamiltonian systems having an arbitrary number of degrees of freedom [18]. The details of the proof were provided in a 1963 paper by Arnol'd, a student of Kolmogorov [1]. Jürgen Moser made an important contribution to the theory by showing the system need only be sufficiently differentiable, rather than analytic, in two dimensions [21, 22]. It is Moser's contribution to KAM theory—known as the *twist theorem*—that is discussed here, with Woodard's simple model in mind.

The twist maps appearing in KAM theory first arose in H. Poincaré's study of a simplified 3-body problem, in which the positions and velocities of 3 point masses in the plane evolve under Newton's inverse square law of gravitation [24]. To place the 3-body problem in the context of this article, imagine one large stationary mass (the sun), and two smaller, uncoupled masses (planets) in periodic motion about the sun (think of two uncoupled 2-body problems). Introducing a small gravitational coupling term between the planets then leads to each planet forcing the other's motion, yielding a small perturbation of a Hamiltonian system.

In the following sections I will explain what twist maps are and illustrate how they arise in a periodically forced nonlinear oscillator in the absence of dissipation. The persistence of invariant simple closed curves in the perturbed system is, remarkably, connected to the Diophantine approximation of irrational numbers, as will be revealed. Moser's KAM theorem will be presented and used to analyze the extraordinarily intricate dynamics exhibited by our simple model. In particular, simulations will be presented to illuminate the complex coexistence of regular and chaotic behaviors. I will conclude by briefly placing this story in the context of Poincaré's pioneering work on the n-body problem and the quest to determine the stability of the solar system.

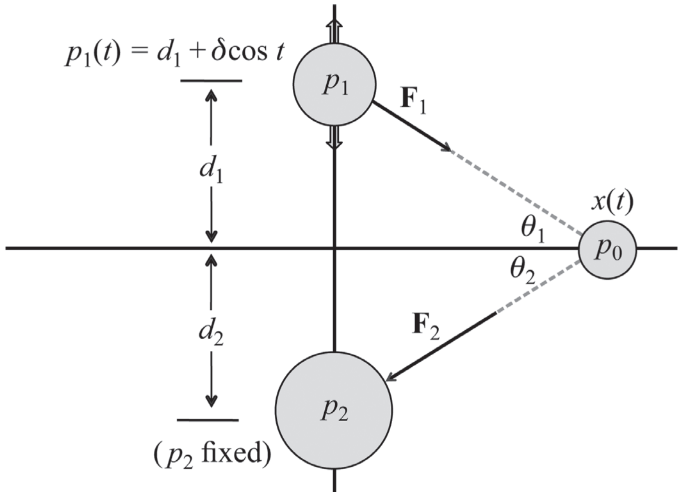

Figure 1 The set-up of the model.

Model equations and the Poincaré map

The set-up The model involves the interaction of three charged particles arranged in the plane in a simple manner. Place a charged particle p_0 of mass m on the x-axis, and two charged particles p_1 and p_2 on the y-axis, straddling the x-axis, as shown in Figure 1. Assume particle p_2, which exerts an attractive force on p_0, remains fixed for all time at a distance d_2 from the x-axis. Particle p_1 exerts a repulsive force on p_0 and is allowed to move vertically, with the distance from p_1 to the x-axis given by the expression

$$p_1(t) = d_1 + \delta \cos t,$$

with $\delta, d_1 > 0$. Particle p_0 is forever constrained to move along the x-axis, with no friction or damping of any kind. To avoid collisions, we take $\delta < d_1$.

The magnitude of the force $\mathbf{F}_i$ exerted on p_0 by particle p_i is assumed to be inversely proportional to the square of the distance r_i from p_0 to p_i, $i = 1, 2$. We also assume $\|\mathbf{F}_2\| > \|\mathbf{F}_1\|$ when p_1 and p_2 are equidistant from the x-axis. Note, however, that if p_0 happens to be near 0 when t is near an odd multiple of π—so that p_1 is close to the x-axis—the repulsive force exerted by particle p_1 on p_0 may dominate the attractive force exerted by p_2. The model results in a type of charged 3-body problem. For the study of a more complex charged 3-body problem, see Atela and McLachlan [4].

Let $x = x(t)$ denote the position of p_0 at time t. Referring to Figure 1 and using model assumptions, we have

$$\|\mathbf{F}_1\| = \frac{c_1}{r_1^2} = \frac{c_1}{(d_1 + \delta \cos t)^2 + x^2}$$

and

$$\|\mathbf{F}_2\| = \frac{c_2}{r_2^2} = \frac{c_2}{d_2^2 + x^2}, \quad 0 < c_1 < c_2.$$

The sum of the horizontal components of $\mathbf{F}_1$ and $\mathbf{F}_2$ is then

$$\begin{aligned} F_x &= \|\mathbf{F}_1\| \cos\theta_1 - \|\mathbf{F}_2\| \cos\theta_2 \\ &= \frac{c_1 x}{((d_1 + \delta \cos t)^2 + x^2)^{3/2}} - \frac{c_2 x}{(d_2^2 + x^2)^{3/2}}. \end{aligned}$$

Via Newton's second law we have $mx'' = F_x$ or, letting $k_i = c_i/m$, $i = 1, 2$,

$$x'' = \frac{k_1 x}{((d_1 + \delta \cos t)^2 + x^2)^{3/2}} - \frac{k_2 x}{(d_2^2 + x^2)^{3/2}}, \quad 0 < k_1 < k_2. \tag{1}$$

The goal is to understand the behavior of solutions to (1) and the way in which this behavior may depend upon the size of the forcing amplitude δ. While a bit sophisticated for the first ODE course, equations such as (1) are ideally suited for junior-level dynamical systems and mathematical modeling courses.

The unforced case To simplify matters, let's assume $d_1 = d_2 = d$ in all that follows. If the forcing is turned off ($\delta = 0$), equation (1) becomes

$$x'' = -\frac{(k_2 - k_1)x}{(d^2 + x^2)^{3/2}}. \tag{2}$$

Note that in a small neighborhood of 0, in which x^2 is much smaller than $|x|$, equation (2) can be approximated by

$$x'' = -\omega^2 x, \quad \omega^2 = \frac{k_2 - k_1}{d^3}, \tag{3}$$

which is the ODE for an undamped harmonic oscillator (see Figure 2(a)).

To see that (2) admits periodic solutions about the origin as well, it is convenient to convert (2) to the first order system

$$\begin{aligned} x' &= y \\ y' &= -\frac{(k_2 - k_1)x}{(d^2 + x^2)^{3/2}} \end{aligned} \tag{4}$$

by letting $y = y(t)$ denote the velocity of particle p_0. The chain rule gives $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$, or

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{(k_2 - k_1)x}{(d^2 + x^2)^{3/2}} \cdot \frac{1}{y} \tag{5}$$

which is, fortuitously, a separable first order equation. The interested reader may verify that

$$\tfrac{1}{2}y^2 - (k_2 - k_1)(d^2 + x^2)^{-1/2} = c, \quad c \in \mathbb{R},$$

is the general solution of (5). Hence solutions of (4) lie on level sets of the function

$$H : \mathbb{R}^2 \to \mathbb{R}, \quad H(x, y) = \tfrac{1}{2}y^2 - (k_2 - k_1)(d^2 + x^2)^{-1/2}.$$

The mapping H, called a *Hamiltonian function*, represents a conserved quantity in the sense that H is constant along a given solution curve (the constant depending on the initial condition). In applications to mechanics, H often represents total mechanical energy (see §5.3 in Blanchard, Devaney, and Hall [**5**]).

Level sets of H are shown in Figure 2(b). Note that the equilibrium solution $(x, y) = (0, 0)$ is encircled by periodic orbits. Indeed, the function H can be used to show that any solution of system (4) with initial velocity $y(0) = 0$ is a periodic solution. In terms of the model this is easily understood: If p_0 starts at $x(0) = x_0 > 0$ with zero velocity it will move to the left, given that $\|\mathbf{F}_2\| > \|\mathbf{F}_1\|$. The velocity of p_0 will hit zero when p_0 reaches position $-x_0$ since level sets of H are symmetric about the y-axis. The particle p_0 then moves to the right, returning to position x_0. A sufficiently large initial velocity *can* overcome the attractive force $\mathbf{F}_2$, with $x(t)$ monotonically approaching $-\infty$ over time in this case. The focus is placed on the

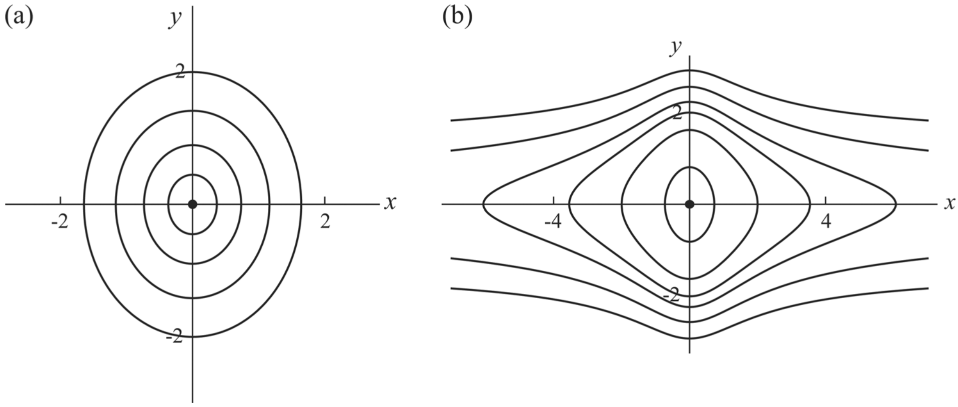

Figure 2 (a) The phase plane for the unforced linear approximation (3) ($y(t) = x'(t)$). (b) The phase plane for the unforced nonlinear system (4). In each case periodic trajectories are moving clockwise.

simple closed curves in Figure 2(b), however, as it is the persistence of a subset of these curves under perturbation that is addressed by KAM theory.

Model equation (1) can thus be solved when the amplitude δ of the forcing function $p_1(t) = d + \delta \cos t$ is set to 0, given the existence of the Hamiltonian function H. The "perturbation" arises via the inclusion of the periodic forcing term for small δ-values.

The model with forcing To begin the analysis for $\delta > 0$, consider the system

$$x' = y \tag{6}$$
$$y' = \frac{k_1 x}{((d + \delta \cos t)^2 + x^2)^{3/2}} - \frac{k_2 x}{(d^2 + x^2)^{3/2}}.$$

The addition of external forcing leads to a nonautonomous system, with solutions living in (x, y, t)-space. Note, however, that the right-hand side of (6) is periodic in t with period 2π. Hence we will sample solutions of (6) at every integer multiple of 2π or, equivalently, each time p_1 returns to its initial position (in essence, identifying the $t = 0$ and $t = 2\pi$-planes; see §5.6 in Blanchard, Devaney, and Hall [**5**] for a nice introduction to this technique). This "stroboscopic" approach leads to consideration of a *Poincaré map*

$$P_\delta : \mathbb{R}^2 \to \mathbb{R}^2, \;\; P_\delta(x_0, y_0) = (x(2\pi), y(2\pi)), \tag{7}$$

where $(x(t), y(t))$ is the solution of (6) for which $(x(0), y(0)) = (x_0, y_0)$. For example, the function $(x(t), y(t)) = (0, 0)$ for all t is a solution of (6); for this solution $P_\delta(0, 0) = (0, 0)$, so that $(0, 0)$ becomes a *fixed point* of P_δ. Generally,

$$P_\delta^n(x_0, y_0) = (x(2\pi n), y(2\pi n)),$$

where P_δ^n denotes n-fold composition of P_δ. The set

$$\{P_\delta^n(x_0, y_0) : n \geq 0\}$$

is the P_δ-*orbit* of (x_0, y_0). The collection of P_δ-orbits will provide insight into the behavior of solutions to system (6).

When $\delta = 0$, the Poincaré map (7) is an example of an *area preserving monotone twist map*, the focus of KAM theory in two dimensions. Just what these terms mean is the substance of the following section.

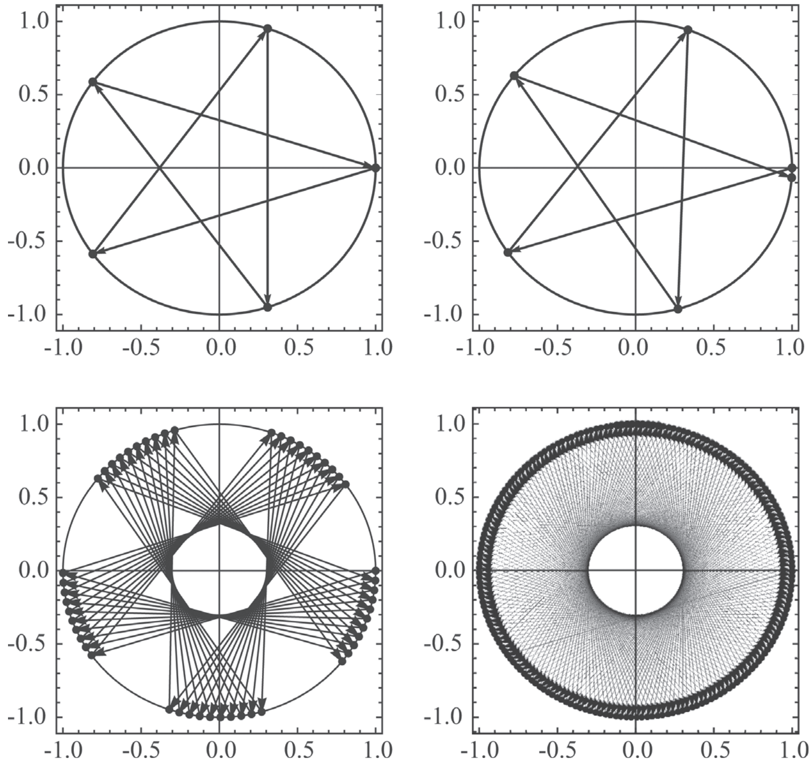

Figure 3 (a) Clockwise rotation by $\frac{2}{5}$. Note 5 is the period of the cycle, while 2 counts the number around the circle each cycle. (b–d) Clockwise rotation by $\frac{2}{5} + \frac{\sqrt{2}}{1000}$, with an increasing number of iterations.

Area preserving twist maps

The twist Recall that the equilibrium point (0, 0) is enclosed by periodic solutions of system (6) when $\delta = 0$ (Figure 2(b)). Each of these simple closed curves γ is *invariant* under the Poincaré map P_0: If $q \in \gamma$, then $P_0(q) \in \gamma$. We can say much more, however. Assume the period T of the solution corresponding to γ satisfies $2\pi/T = m/n \in \mathbb{Q}$, where m/n is in lowest terms. Applying P_0 n times to any point $q \in \gamma$ generates an orbit that goes m times around γ before returning to q. In particular, each $q \in \gamma$ becomes a *periodic point of period n* for P_0, with the average "rotation" around γ per iterate of P_0 given by m/n (see Figure 3(a)). This average rotation per iterate of P_0 on the invariant curve γ is known as a *rotation number*, an important topic in the study of the dynamics of circle maps (see Walsh [**28**] for an in-depth introduction to rotation numbers).

On the other hand, it's not too hard to show the P_0-orbit of q is an infinite set if $2\pi/T \notin \mathbb{Q}$. It is more of a challenge to show the P_0-orbit of q is dense in γ in this case [**28**] (see Figure 3(b–d)). The average rotation of P_0 per iterate along the invariant curve γ is the irrational $2\pi/T$. The dichotomy between the behaviors associated with rational and irrational rotation numbers is clearly evident when plotting P_0-orbits (see Figure 4).

The P_0-rotation number on the invariant curve γ is plotted as a function of the distance r from the origin to the intersection of γ with the x-axis in Figure 5. This was accomplished by converting (4) into polar coordinates and computing the average angular change per iterate of P_0. The values plotted in Figure 5 are negative as

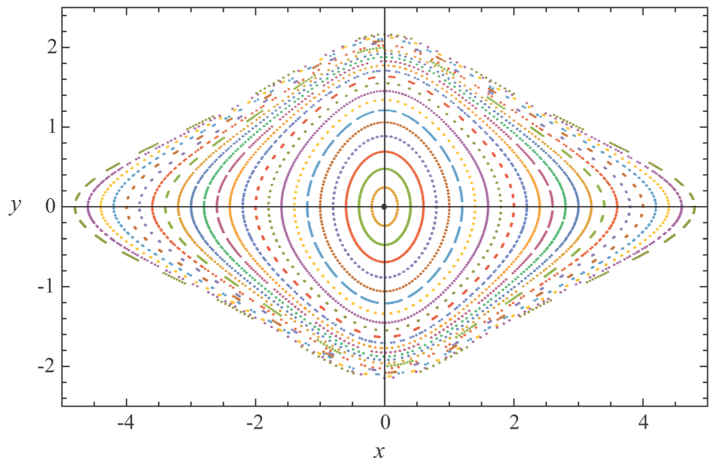

Figure 4 Iteration of the Poincaré map P_0 in the absence of forcing.

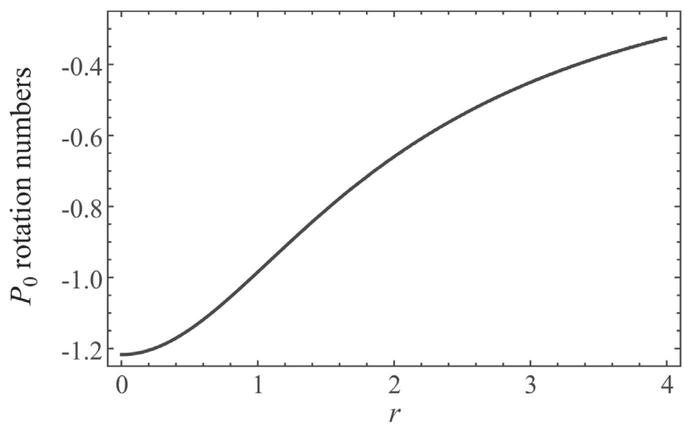

Figure 5 The rotation number for P_0 when restricted to the invariant curves in Figure 4 as a function of the distance r to the origin.

P_0-orbits are rotating clockwise. That the absolute value of this function is monotone decreasing arises from the fact that the period of a (periodic) solution of system (4) increases with r, so that increasing numbers of iterates of P_0 are needed to cycle once around a given curve γ.

We can now see why P_0 is referred to as a monotone twist map. Note that two simple closed P_0-invariant curves bound an annular region. A twist map is, generally, a mapping of an annulus that rotates the inner and outer boundary at different rates, as is the case for P_0 on *any* two invariant curves in Figure 4 (recall the monotone nature of the P_0-rotation plot).

We can again say much more, however. Consider three P_0-invariant curves as in Figure 6(a), with the middle curve γ_2 having rotation number $m/n \in \mathbb{Q}$. Let $\alpha_1 < \alpha_3$ denote the rotation numbers of P_0 when restricted to the inner and outer curves γ_1 and γ_3, respectively (recall $|\alpha_1| > |\alpha_3|$). Let q_i denote the intersection of a ray extending from the origin with $\gamma_i,\ i = 1, 2, 3$; note $P_0^n(q_2) = q_2$ as q_2 has period n under P_0.

Given $|\alpha_1| > |\alpha_2|$, $P_0^n(q_1)$ will have advanced clockwise beyond $P_0^n(q_2)$, while $P_0^n(q_3)$ will have fallen behind (moved counterclockwise) relative to $P_0^n(q_2)$. The annular region bounded by γ_1 and γ_3 has the property that the inside boundary curve

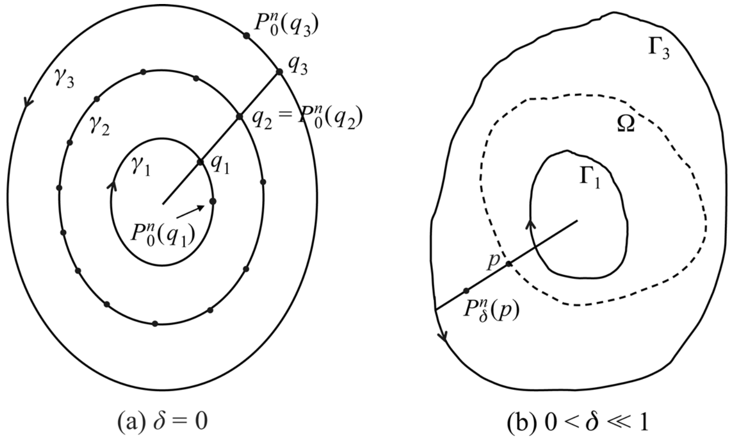

Figure 6 (a) P_0 has rotation number α_i on γ_i, with α_1 and α_3 Diophantine irrationals and $\alpha_2 = \frac{m}{n}$. (b) By the twist theorem, P_δ has rotation number α_i on the invariant curve Γ_i, $i = 1, 3$. The dashed curve Ω consists of points p for which p and $P_\delta^n(p)$ have the same polar angle.

turns clockwise, while the outside boundary curve turns counterclockwise, under P_0^n. This behavior is a manifestation of the twisting action of P_0. Before describing the role played by this behavior in understanding the dynamics of P_δ with $\delta > 0$, we must pause to discuss the second crucial aspect of KAM theory in two dimensions. Namely, area preserving maps.

The Poincaré map P_δ preserves area Consider the vector field

$$\mathbf{V} : \mathbb{R}^2 \to \mathbb{R}^2, \quad \mathbf{V}(x, y) = \left(y, -\frac{(k_2 - k_1)x}{(d^2 + x^2)^{3/2}} \right) \tag{8}$$

corresponding to (4) (and to system (6) when $\delta = 0$). Given $(x_0, y_0) \in \mathbb{R}^2$, let

$$\phi_t(x_0, y_0) = (x(t), y(t))$$

denote the solution to (4) satisfying the initial condition $(x(0), y(0)) = (x_0, y_0)$. Thinking of $\phi_t(x_0, y_0)$ as a function of both t and (x_0, y_0) yields what is called the *flow* associated to (4).

Since the divergence of $\mathbf{V}$ satisfies $\operatorname{div} \mathbf{V} = 0$, areas in the xy-plane are preserved under the flow ϕ_t, a consequence of Liouville's formula (see Hartman [**14**, p. 46]). This implies in particular that the Poincaré map P_0 preserves area: for a domain $D \subset \mathbb{R}^2$, $\text{area}(P_0(D)) = \text{area}(D)$ (see Figure 7). Hence P_0, as well as all iterates of P_0, can have neither attracting nor repelling fixed points, since sufficiently small neighborhoods of the fixed point are contracted in the former case and dilated in the latter case [**3**]. Fixed points enclosed by invariant simple closed curves, as is the origin in Figure 4, are of *elliptic* type. Fixed points with one stable and one unstable direction—the linearization of the vector field at the fixed point has eigenvalues satisfying $|\lambda_1| < 1 < |\lambda_2|$—are called *saddles*. (For area preserving maps, $\lambda_2 = 1/\lambda_1$ in the saddle case [**22**].) Elliptic and saddle fixed and periodic points *can* occur in area preserving mappings.

The Poincaré map P_δ can be viewed as a perturbation of P_0 for small δ-values, as discussed above. KAM theory applies when the perturbed map P_δ retains the characteristic of area preservation exhibited by P_0. That this holds here follows from the following proposition:

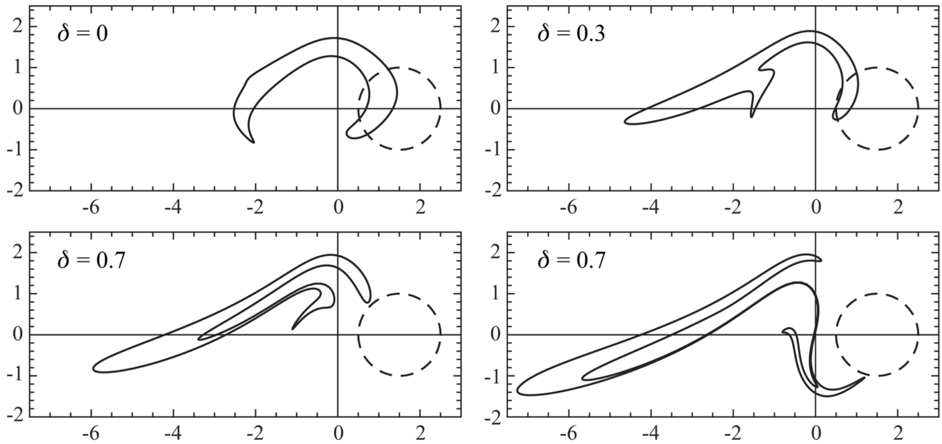

Figure 7 The map P_δ preserves area for $\delta \geq 0$. In each plot P_δ maps the disk bounded by the dotted circle to the set of points interior to the solid curve ($d = 1.5, k_1 = 5, k_2 = 10$).

Proposition 1. *For* $\delta > 0$, $P_\delta : \mathbb{R}^2 \to \mathbb{R}^2$ *preserves area.*

Proof. Let $\delta > 0$. Resorting to a bit of standard trickery by setting $z = t$, convert (6) into the autonomous system

$$\begin{aligned} x' &= y = f_1(x, y, z) \\ y' &= \frac{k_1 x}{((d + \delta \cos z)^2 + x^2)^{3/2}} - \frac{k_2 x}{(d^2 + x^2)^{3/2}} = f_2(x, y, z) \\ z' &= 1 = f_3(x, y, z). \end{aligned} \tag{9}$$

The vector field $F = (f_1, f_2, f_3)$ has divergence 0, so that volumes in $\mathbb{R}^3$ are preserved under the flow associated to (9). Now, consider a domain D in the $t = 0$-plane, and let W denote the solid with base D and height Δt. Let W' denote the image of W under the flow after time 2π. Note

$$\text{Vol}(W') = \text{Vol}(W) = \text{area}(D)\Delta t.$$

For $0 \leq s \leq \Delta t$, let $a(s)$ denote the area of the intersection of W' with the $t = (s + 2\pi)$-plane. By the fundamental theorem of calculus, we have

$$\begin{aligned} \text{area}(P_\delta(D)) = a(0) &= \lim_{\Delta t \to 0} \frac{\int_0^{\Delta t} a(s)ds}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\text{Vol}(W')}{\Delta t} \\ &= \text{area}(D). \end{aligned}$$

■

We see that the study of the forced, nonlinear oscillator modeled by (6) reduces to the study of iteration of the area preserving mapping (7), which for $\delta = 0$ is a monotone twist map on the annular region bounded by any two P_0-invariant curves. This is precisely the topic addressed by Moser's contribution to KAM theory.

Moser's twist theorem

Recall that each point $q \in \gamma$, where γ is a P_0-invariant curve for which the rotation number is $m/n \in \mathbb{Q}$, satisfies $P_0^n(q) = q$. It would seem plausible that a curve of points exhibiting such exceptionally regular behavior would not persist under perturbation. In other words, it would not be surprising if there were no P_δ-invariant curve γ' on which all P_δ-orbits were periodic with period n and rotation number m/n, even for small δ-values. If the rotation number α for P_0 on γ is irrational, then P_0 has no periodic orbits on γ as each P_0-orbit is dense in γ. Nonetheless, intuition might suggest the fact that γ is a topological circle would not persist, so that P_δ need not have an invariant curve γ' homeomorphic to γ and on which all P_δ-orbits were dense with rotation number α.

Remarkably, for α irrational P_δ *will* have just such a curve γ', provided the amplitude δ of the forcing is sufficiently small and α satisfies a *Diophantine condition*. This latter requirement, informally, says that the irrational α is *poorly approximated by rationals*.

Diophantine approximation The approximation of irrationals by rationals has a rich history in number theory, with strong connections to topics such as continued fractions and the theory of transcendental numbers. I will refer the reader to Burger [6] for an engaging and hands-on introduction to the subject, quoting but a few beautiful results here before moving on with our model analysis.

For an irrational α, one might ask if there exist constants N and C such that the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{C}{q^N}$$

holds for infinitely many relatively prime integers p and q. If yes, then there are rationals approaching α at a certain rate, depending on the denominators of these rationals to the power N. For example, a consequence of P. Dirichlet's 1842 theorem on Diophantine approximation is that for algebraic irrationals α,

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

has infinitely many solutions in coprime integers p and q [7]. A full century later, K. Roth showed the exponent 2 is optimal by proving that for an algebraic irrational α and $\epsilon > 0$,

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$$

has only *finitely* many solutions [25].

A half century after Dirichlet's result, A. Hurwitz showed that for *any* irrational α,

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$$

always admits infinitely many solutions in coprime integers p and q [17]. Moreover, he proved the constant $\sqrt{5}$ cannot be increased: if $K > \sqrt{5}$ and $\alpha = \frac{1}{2}(1 + \sqrt{5})$ is the golden mean, then

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Kq^2}$$

has only finitely many solutions. Again, the interested reader is urged to consult Burger [6] and the references therein for many more results in this fascinating area of number theory.

The theorem Returning to our model, and with an eye toward Moser's theorem, an irrational α satisfying

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C}{|q|^{5/2}}, \tag{10}$$

for some constant $C = C(\alpha)$ and all coprime integers p and q, will be said to satisfy a *Diophantine condition*. Such an α is also said to be *poorly approximated by rationals*. Let $\mathcal{D}$ denote the set of all such irrationals α (a set of full measure if $5/2$ in (10) is replaced by any $\mu > 2$ [2]). It is the P_0-invariant simple closed curves γ on which P_0 has a rotation number $\alpha \in \mathcal{D}$ that persist under the area preserving perturbation.

In the following version of Moser's twist theorem (see Hale and Koçak [13, §15.6]), chosen for the relative simplicity of its statement), A is an annulus given in polar coordinates by

$$A = \{(r, \theta) : a \leq r \leq b\}.$$

For each $c \in [a, b]$ the circle $r = c$ is invariant under f_0, with f_0 having rotation number $\eta(c)$ on this circle. The condition $|\eta'(r)| > 0$ ensures f_0 is a monotone twist map. The use of a map of the form f_0 stems from the fact that a coordinate change exists in which the P_0-invariant curves become circles, and the rotation on a given circle is through an angle of the form $2\pi\eta(r)$ as in $f_0(r, \theta)$ (see Arrowsmith and Place [3, §1.9]). The map g is periodic in θ with period 2π.

Theorem 1. *Consider the area preserving perturbation of a twist map*

$$f_\delta : A \to A, \quad f_\delta(r, \theta) = f_0(r, \theta) + \delta g(\delta, r, \theta),$$

where

$$f_0(r, \theta) = (r, \theta + 2\pi\eta(r))$$

and $|\delta|$ *is sufficiently small. Assume that* g *is* C^5 *and* $|\eta'(r)| > 0$. *For any* α *in both* $\mathcal{D}$ *and the range of* $\eta(r)$, *there exists a simple closed curve* Γ *that is invariant under* f_δ *and on which* f_δ*-orbits are dense. Furthermore,* f_δ *has rotation number* α *on* Γ.

Remark. (a) Note this result holds for *every such irrational number* α. Moser's proof additionally shows the Lebesgue measure of the set of f_δ-invariant curves is positive and approaches the measure of the annulus as $\delta \to 0$.

(b) Number theory plays an important role in other areas of dynamical systems such as ergodic theory [9]. The interplay of dynamical systems and number theory is central to the relatively new field of arithmetic dynamics [26].

(c) In rigorous proofs of KAM theorems, the bounds on the analog of the above perturbation parameter δ are often incredibly small. In simulations, however, invariant curves and quasiperiodic motion are typically seen to persist for much larger values of the perturbation parameter (see Dumas [11, §4.7]). In the model presented here, the invariant curves completely disappear at $\delta \approx 0.286$. This "threshold" depends upon the relative sizes of $\|\mathbf{F}_1\|$ and $\|\mathbf{F}_2\|$, that is, upon the relative masses of p_1 and p_2.

The reader may be puzzled by the requirement that α be poorly approximated by rationals. This hypothesis stems from the *small divisor problem*, to which a nice introduction is presented by Dumas [11] (also see Arrowsmith and Place[3], and Wayne

[29]). To say just a few words here on this formidable problem, one seeks to solve equations of the form

$$j(\tilde{\theta} + \alpha) - j(\tilde{\theta}) = h(\tilde{\theta}), \tag{11}$$

where h is periodic with period 1 (here $\tilde{\theta} = \theta/2\pi$), in trying to show there is a parameterization of an invariant simple closed curve for the perturbed function. Fourier series are used to seek formal solutions of (11). To wit, express

$$j(\theta) = \sum_{k\in\mathbb{Z}} a_k \exp(i2\pi k\theta)$$

and

$$h(\theta) = \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} b_k \exp(i2\pi k\theta)$$

(for technical reasons, $k \neq 0$ in the expansion for h (see Dumas [11, §3.11]). Substituting into (11) and comparing coefficients yields

$$a_k = \frac{b_k}{\exp(i2\pi k\alpha) - 1}, \quad k \in \mathbb{Z},\ k \neq 0. \tag{12}$$

This process breaks down for any rational α since the denominator in (12) would hit 0 in this case. KAM theory thus says nothing about the persistence of invariant curves having rational rotation number. For α irrational, the expression $\exp(i2\pi k\alpha) - 1$ never equals 0. However, this expression cannot become *too small relative to* $|b_k|$ if the series for j is to converge. This is where the assumption that α is poorly approximated by rationals comes into play.

Consequences

Returning to the set-up depicted in Figure 6, choose any two P_0-invariant curves γ_1 and γ_3 having Diophantine irrational rotation numbers α_1 and α_3, respectively. By Moser's theorem, there exist P_δ-invariant curves Γ_1 and Γ_3 on which P_δ has rotation numbers α_1 and α_3, provided the amplitude δ of the forcing is sufficiently small. Pick any rational $\frac{m}{n} \in (\alpha_1, \alpha_3)$. Recall that P_0^n moves points on γ_1 clockwise, relative to radially corresponding points on γ_2, while points on γ_3 move counterclockwise under P_0^n. If δ is small enough, this relative behavior will persist on Γ_1 and Γ_3 under P_δ^n. By continuity, on each ray extending from the origin and intersecting Γ_1 and Γ_3 there is a point p with the property that p and $P_\delta^n(p)$ have the same polar angle. Let Ω denote the collection of such points p as the ray turns through 2π radians.

What can be said about the image $\Omega' = P_\delta^n(\Omega)$? Since P_δ preserves area, each of Ω and Ω' enclose the same area. Hence it cannot be the case that one of these curves encircles the other, leading to the conclusion that Ω and Ω' intersect in an even number of points (for a typical perturbation Ω and Ω' intersect transversally, as in Figure 8). For $p \in \Omega \cap \Omega'$ each of p and $P_\delta^n(p)$ have the same polar angle and radius, implying $P_\delta^n(p) = p$. Each point of intersection of Ω and Ω' is thus a periodic point of period n for P_δ. Moreover, this argument can be made *for every rational* $\frac{m}{n} \in (\alpha_1, \alpha_2)$, and for *every pair of Diophantine irrationals* realized as rotation numbers for P_0.

Recall that fixed and periodic points of P_δ must be of either elliptic or saddle type, because P_δ preserves area. The relative clockwise motion of P_δ^n on Γ_1 and

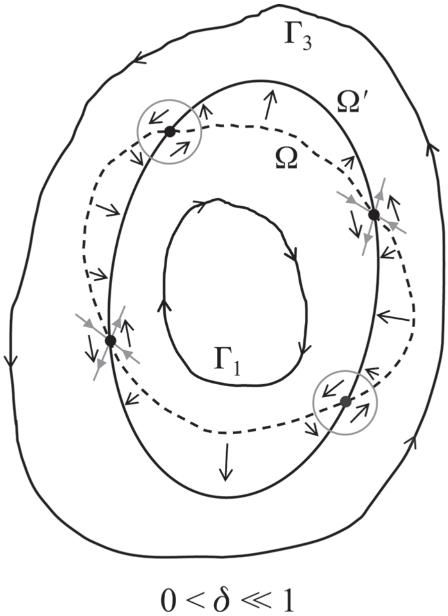

Figure 8 The intersection of Ω and Ω' yields fixed points of P_δ^n. The arrows represent the direction of orbits under P_δ^n. Note P_δ^n-orbits circulate around in a neighborhood of the top left and bottom right (elliptic) fixed points, while P_δ^n-orbits leave small neighborhoods of the top right and bottom left (saddle) fixed points.

counterclockwise motion on Γ_3, together with the positioning of Ω' relative to Ω, indicates the periodic points of P_δ are alternatively of elliptic and saddle type (see Figure 8). At this point I would ask the reader to take a moment and attempt to visualize just what a plot of P_δ-orbits in the xy-plane would look like. A beautifully complicated image to be sure!

As with any good story, however, there is much more to the picture. Note each elliptic periodic point of period n for P_δ is an elliptic fixed point of the mapping $h_\delta = P_\delta^n$. Thus, the entire analysis to this point can be repeated in a neighborhood of every elliptic periodic point of P_δ. The behavior of the P_δ-orbits you were trying to envision occurs infinitely often on ever smaller scales, with chains of saddle and elliptic periodic points surrounding each and every elliptic periodic point.

Simulations The amazing behavior exhibited by Woodard's simple model can be simulated with a tool such as *Mathematica*. For δ very close to zero, one would expect to see only the invariant curves with Diophantine rotation numbers. As δ increases, these invariant curves should begin to break up, with periodic and chaotic behavior becoming easier to discern, behavior that is nested on ever smaller scales.

In each of Figures 9–11, the values $d = 1.5$, $k_1 = 5$ and $k_2 = 10$ were used. Plotted in Figure 9 are P_δ orbits for $\delta = 0.001, 0.07$ and 0.11. Note the loss of invariant curves as δ increases, with a corresponding increase in chaotic orbits. Zooming in when $\delta = 0.11$, a period 10 elliptic-saddle chain is clearly evident in the bottom right plot (period $10 = 2 \times 5$ due to the symmetry about the y-axis; 5 equaling the number of elliptic (and saddle) periodic points in the largest visible chain). A close look also reveals periodic points of period $22 = 2 \times 11$ (sitting just outside the invariant closed curves). The point in the center of the "bullseye"—near $x = 2.7$—is a periodic point of period 2.

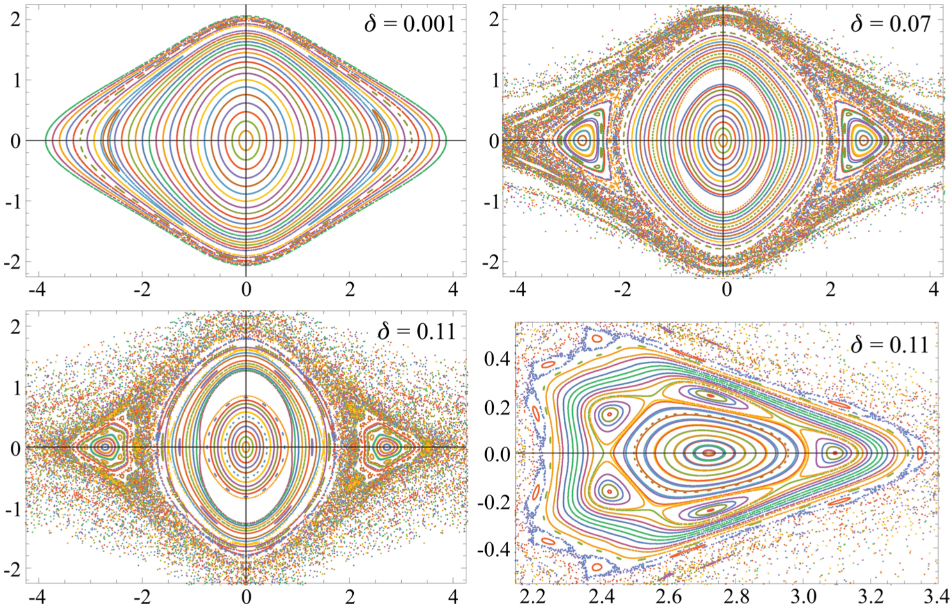

Figure 9 P_δ-orbits for increasing δ. Invariant curves are lost, while chaotic orbits become more prevalent, as δ increases. Initial conditions of the form $(x_0, 0)$ were used, $x_0 \in [0, 4]$.

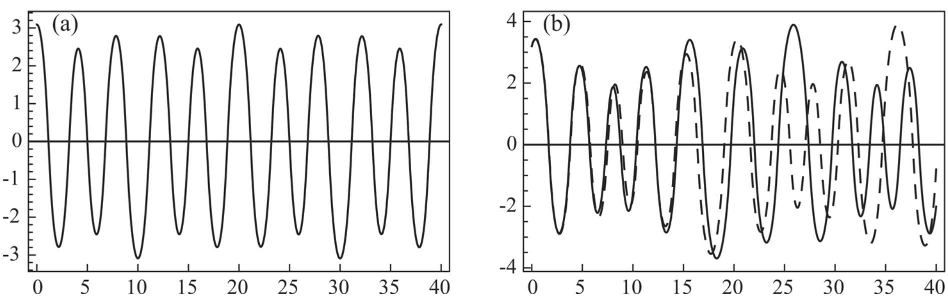

Figure 10 The position of p_0 as a function of time for $\delta = 0.11$. (a) $x(t)$ for the period-10 orbit visible in the bottom right image in Figure 9. (b) $x(t)$ plots for two almost identical initial conditions chosen in the chaotic region in Figure 9, illustrating sensitive dependence on initial conditions for $P_{0.11}$.

In Figure 10(a), the $x(t)$-values corresponding to the period 10 orbit of $P_{0.11}$ are plotted (note $x(t + 10 \cdot 2\pi) = x(t)$). The fact $P_{0.11}$ also exhibits sensitive dependence on initial conditions in certain bounded regions of the plane is illustrated in Figure 10(b), where the initial conditions used are (3.2, 0.4) and (3.2, 0.41).

Increasing the amplitude of the forcing brings repelling particle p_1 ever closer to the x-axis and particle p_0. Very few invariant curves remain when $\delta = 0.15$ or 0.19, although periodic orbits continue to exist (Figure 11).

Concluding remarks

Having no access to computer simulations, Poincaré nonetheless realized—quite remarkably—that what are now known as “twist maps” and “chaotic behavior” appear in the 3-body problem [24]. While he showed that the 3-body problem was not

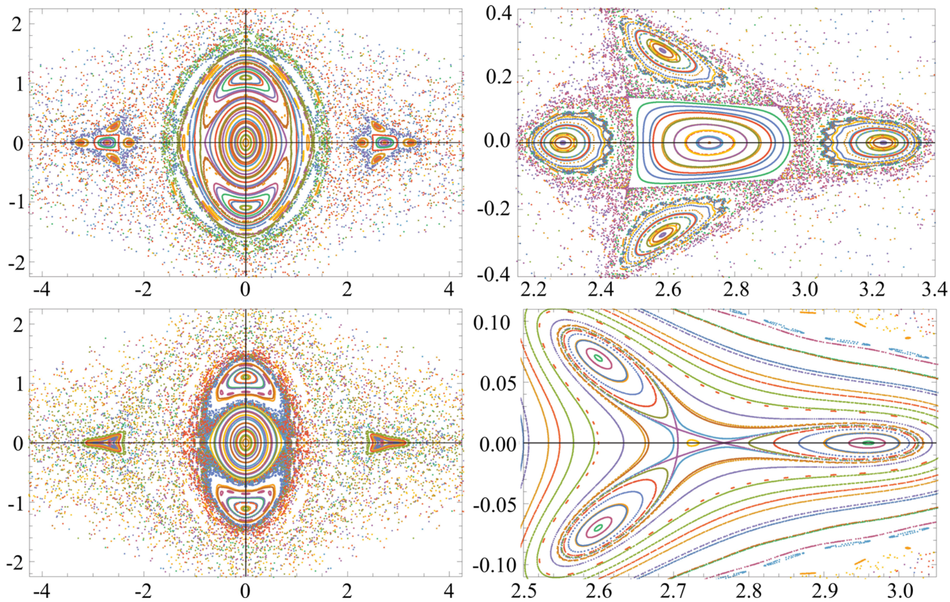

Figure 11 P_δ-orbits. *Top row*: $\delta = 0.15$. *Bottom row*: $\delta = 0.19$.

completely integrable (in the sense of system (4)), Poincaré was unable to prove the existence of invariant curves for his analog of model map P_δ.

That the work of Kolmogorov, Arnol'd, and Moser might be interpreted as providing results concerning *stability* is easily seen through the lens of Woodard's model. Note that solutions to system (6) starting at

$$(x, y, 0) \in \Gamma \times \{0\}$$

in the $t = 0$-plane, where Γ is a P_δ-invariant curve, intersect the curve $\Gamma \times \{2\pi\}$ in the $t = 2\pi$-plane. Since the model equations (6) are periodic in t with period 2π, the $t = 0$ and $t = 2\pi$-planes can be identified. The identification of $\Gamma \times \{0\}$ with $\Gamma \times \{2\pi\}$ leads to solutions that lie on an invariant torus (with each solution curve dense in this torus). By the existence and uniqueness theorem for ODE, any solution starting between invariant curves Γ_1 and Γ_2 will remain trapped between the invariant tori corresponding to Γ_1 and Γ_2 for all time. Such invariant tori were present in Poincaré's analysis of the 3-body problem. As in Woodard's model, stability for the restricted class of 3-body problems Poincaré was investigating follows from KAM theory.

As with many mathematicians before and since, Poincaré was motivated by the quest to determine if the solar system is stable. Will the planets maintain their current configurations, more or less, for evermore? Or might one planet crash into another, or into the sun, or perhaps leave the solar system entirely, at some future time? The mathematical treatment of this question goes back to Newton's formulation of the n-body problem: determine the evolution of n point masses in space, interacting under the inverse square law of gravitation, for all past and for all future times. The governing equations, which are Hamiltonian in nature, can be recast so as to fall within the realm of KAM theory [**11**, **22**].

Of course, the n-body problem is but a mathematical model of our solar system, so stability in the n-body problem is not the same as stability of our solar system. In the mathematical treatment of the n-body problem, the perturbation parameter is related to the ratio of the planetary masses to the mass of the sun, and for our solar system this ratio is too large to allow for direct application of KAM theory [11]. Numerical work carried out over very long time scales does indicate that the behavior of the inner planets—Mercury, Venus, Earth, and Mars—is chaotic [19, 20, 27], so that perhaps our solar system would sit outside any regions of stability in any event.

Applications of KAM theory to physical models has been aided by the development of computer assisted proof techniques. The stability question can be investigated by long computations on a computer in which machine rounding errors are rigorously tracked. Interactions of KAM theory and computer assisted proofs have been successfully applied to various 3-body problems (the sun, Jupiter and Saturn, by Chelletti and Chierchia [8], for example), and to various charged particle systems such as particle accelerators [12, 21].

One might fairly ask whether the major conceptual advances presented by KAM theory outshine the practical applications of the theory. Moser [23] is clear on his view of this issue, writing in a delightful essay:

> Is the solar system stable? Properly speaking, the answer is still unknown, and yet this question has lead to very deep results which probably are more important than the answer to the original question.

The development of KAM theory stands as one of the major mathematical achievements of the 20th century. With powerful computing capabilities now at our fingertips, the investigation of simple models such as Woodard's presents an enticing entry point into what is truly a wonderful world.

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Summary. What happens when you periodically force a nonlinear oscillator in the absence of damping? For linear oscillators, such as the mass-on-a-spring model typically encountered in that first differential equations course, the behavior of the forced system is easily and well understood. In this article, a simple mechanical model is used to illuminate the power and beauty of the theory of Kolmogorov, Arnol'd, and Moser. Known as KAM theory, this profound 20th century mathematical achievement answers the question posed above. Model simulations illustrating the complex coexistence of regular and chaotic motions are presented. Additionally, KAM theory is placed within its historical context, namely, the quest to determine the stability of the solar system.

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Recognizing Cayley Digraphs

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Visualization is a powerful pedagogical tool for building intuition about abstract mathematical concepts. We draw graphs to convey information about functions. Intersecting planes portray the solution set of a pair of simultaneous linear equations. Venn diagrams reveal ways in which sets interact with one another. While mathematical ideas can be expressed through symbols alone, visual representations rouse our senses toward a richer level of comprehension.

Cayley digraphs

Arthur Cayley [1] proposed an ingenious means of using a directed graph (digraph) to visualize a group G. Let S be a generating set for G. Denote each element $g \in G$ by a vertex $\bullet_g$ and let each member of S be represented by a distinct type of arrow. For example, if $S = \{a, b\}$ we might denote

$$a \text{ as } \longrightarrow, \qquad \text{and} \qquad b \text{ as } - \rightarrow .$$

With this notation, an occurrence of $\bullet_g \longrightarrow \bullet_h$ in our graph will mean $ga = h$, and we will say that edge $(\bullet_g, \bullet_h)$ is of *arrow-type* a. In other words, a directed edge of a given arrow-type indicates right multiplication: the group element represented by the vertex at the start of the arrow, times the generator represented by the arrow-type, is equal to the group element represented by the vertex at the end of the arrow. Alternatively, $\bullet_g \longrightarrow \bullet_h$ will also mean $ha^{-1} = g$, and we will say that edge $(\bullet_h, \bullet_g)$ has arrow-type a^{-1}. In other words, traversing an edge of the graph in the direction opposite that indicated by the arrow corresponds to multiplication on the right by the inverse of the corresponding generator. If a generator $b \in S$ is its own inverse (i.e., if b^2 is the identity element of G), we will represent all edges of arrow-type b without arrowheads.

A digraph representing a group G that is constructed in the manner described above is called a *Cayley digraph* of G.

I use Cayley digraphs extensively in my introductory abstract algebra course in order to visualize groups, and to intuitively illustrate group theoretic results. After examining Cayley digraphs for a number of elementary groups, students are ready to tackle the empowering task of discovering new groups by constructing digraphs of their own. But how do they know if they have succeeded? This raises the motivating question that prompted this article:

Question 1. *Given a digraph Γ with edges of various arrow-types, how can we recognize whether Γ is a Cayley digraph of some group?*

Several conditions on Γ are straightforward. First of all, if Γ is a Cayley digraph of some group G, then (1) we must be able to get from any vertex, v, of Γ to any vertex, w, by traveling along consecutive edges of Γ in the directions of the arrows. This follows

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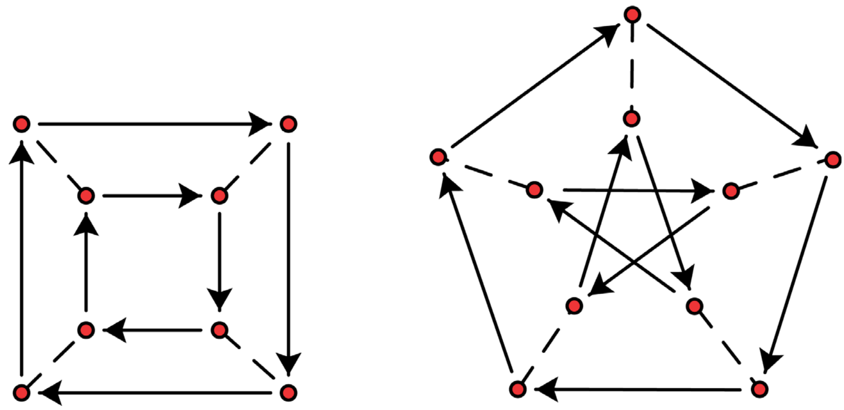

Figure 1 Which of these digraphs is a Cayley digraph of a group?

from the fact that $a * x = b$ must have a solution for every $a, b \in G$. Furthermore, since $a * x$ and $a * x^{-1}$ each have a unique value for each $a, x \in G$, it follows that (2) every vertex v of Γ must have exactly one edge of each arrow-type starting at v, and exactly one edge of each arrow-type ending at v. Finally, since $a * x = a * y$ implies that $x = y$ for all $a, x, y \in G$, we have that (3) at most one edge must connect any given pair of vertices of Γ.

Both of the graphs in Figure 1 satisfy the three conditions. However, only one of these is a Cayley digraph of a group. How can we determine which digraph is Cayley, and which one is not?

Readers familiar with Cayley digraphs might recall the celebrated result of Gert Sabidussi [3] that we state below as Theorem 1. Recall that a *graph automorphism* of a digraph Γ is a bijection on the vertex set $V(\Gamma)$ that preserves arrow types, and that the set of all graph automorphisms of Γ forms a group (denoted $\mathrm{Aut}(\Gamma)$) with respect to composition. Recall also that Γ is *vertex transitive* provided that for any two vertices v_1 and v_2 of Γ, there is a graph automorphism f for which $f(v_1) = v_2$.

Theorem 1 (Sabidussi). *If Γ is any (undirected) finite graph, then Γ can be directed and assigned arrow-types in such a way that the resulting directed graph is isomorphic to a Cayley digraph for some group G if and only if* $\mathrm{Aut}(\Gamma)$ *contains a subgroup H for which (i) $|H| = |V(\Gamma)|$, and (ii) H acts transitively on $V(\Gamma)$. In this case, H is isomorphic to G.*

Sabidussi's theorem was originally presented as a lemma that was used to prove results about undirected graphs. In fact, Sabadussi makes no mention of Cayley graphs in his paper. While Theorem 1 does provide a crisp characterization of Cayley digraphs, its use is clearly impractical in an introductory abstract algebra course.

A starting place

John Fraleigh provides one answer to our motivating question [2, p. 70]. He asserts that a digraph Γ with edges of various arrow-types is a Cayley digraph of some group if and only if the following four conditions are satisfied:

1. We can get from any vertex v to any vertex w by traveling along consecutive edges in the directions of the arrows (i.e., Γ is *connected*).
2. Every vertex v of Γ has exactly one edge of each arrow-type starting at v, and exactly one edge of each arrow-type ending at v (i.e., Γ is *regular*).

3. At most one edge connects any given pair of vertices.
4. If two different sequences of arrow-types that begin at any vertex v both end at the same vertex w, then those same two sequences of arrow-types that begin at any vertex v' must both end at the same vertex w'.

Conditions 1–3 were discussed in the previous section and can be checked in short order for even moderately sized digraphs. However, verifying condition 4 requires much more work. For example, the graph on the left in Figure 1 contains $\binom{8}{2} = 28$ pairs of vertices. Even after accounting for loops, the number of paths that would need to be considered between pairs of vertices in order to check condition 4 would be daunting.

We seek an alternative to condition 4 that is both necessary and sufficient to determine (in the presence of conditions 1–3) whether a digraph of various arrow-types represents a group. Moreover, we want a classroom-friendly condition that is easy to check for moderately sized digraphs. Such a condition will be discovered in the sections which follow.

We end this section by showing directly that the graph on the left in Figure 1 is a Cayley digraph of a group.

Example 1. Consider the sets $\mathbf{Z}_4 = \{0, 1, 2, 3\}$ and $\mathbf{Z}_2 = \{0, 1\}$ together with addition mod 4 and addition mod 2, respectively. The group

$$\mathbf{Z}_4 \times \mathbf{Z}_2 = \{(x, y) : x \in \mathbf{Z}_4, y \in \mathbf{Z}_2\}$$

can be generated by $S = \{(1, 0), (0, 1)\}$. Representing $(1, 0)$ by $\longrightarrow$ and $(0, 1)$ by - - - - (no arrowheads since $(0, 1)$ is its own inverse) yields the Cayley digraph of $\mathbf{Z}_4 \times \mathbf{Z}_2$ illustrated in Figure 2.

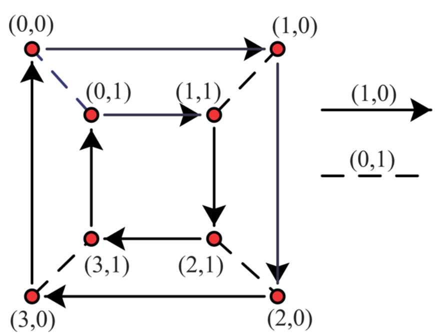

Figure 2 A Cayley digraph representing $\mathbf{Z}_4 \times \mathbf{Z}_2$.

Recovering a group from a Cayley digraph

Note that condition 4 asserts that a Cayley digraph is vertex transitive. In this sense, all of its vertices are identical from a graph theoretic viewpoint; that is, no vertex of a Cayley digraph can be distinguished from any other. In particular, any vertex of a Cayley digraph may be used to represent the identity element of the underlying group. Because of this symmetric structure, the vertices of a Cayley digraph need not be labeled. Clear identification of the arrow-types that are used to form the edges of the graph are all that is needed to completely describe the structure of the group. Therefore, if the vertex identifiers are removed from the graph in Figure 2, the digraph that remains illustrates the isomorphism class that contains the group $\mathbf{Z}_4 \times \mathbf{Z}_2$.

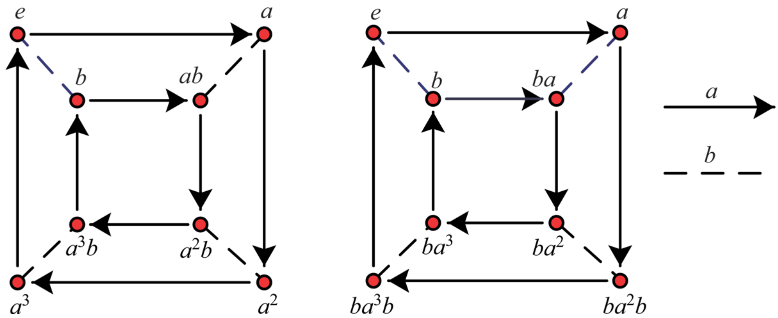

Figure 3 Two labelings of a Cayley digraph representing $\mathbf{Z}_4 \times \mathbf{Z}_2$.

We can recover a group G from an unlabeled Cayley graph Γ by using the arrow-types comprising the edges of the graph to assign labels to each vertex. Begin by choosing a vertex of the graph to represent the identity element of the group. Since the arrow-types used in the graph represent generators of G, each vertex v of the graph can be described by a string of arrow-types, read from left to right, that takes us from the vertex chosen to represent the identity element to the vertex v.

We will call such a string of arrow-types a *label* for v. We will always use e to denote the vertex representing the identity element of G, and we will always use the empty string of arrow-types as the label for e—that is, the label for e will be the string consisting of no arrow-types at all. The collection of all assigned labels will be called a *labeling* of Γ. When λ is used to denote a particular labeling of Γ, we use $\lambda(v)$ to indicate the label assigned to vertex v. Two different labelings for groups in the $\mathbf{Z}_4 \times \mathbf{Z}_2$ isomorphism class are illustrated in Figure 3. Observe that consecutive arrow-types are abbreviated using exponential notation.

When a group G is represented by a Cayley digraph Γ, and a labeling λ is assigned to Γ, the labeling induces a natural operation $*$ on the vertex set $V(\Gamma)$ of Γ defined by

$$v * w = v\lambda(w)$$

where $v\lambda(w)$ is the vertex arrived at when beginning at vertex v and traversing the arrow-types identified in the sequence $\lambda(w)$ from left to right. The resulting algebraic structure consisting of the set $V(\Gamma)$ with operation $*$ is isomorphic to G. (We will use vw to denote $v * w$ from now on.)

This procedure for recovering a group (up to isomorphism) from a Cayley digraph Γ presupposes that Γ actually *is* a Cayley digraph of some group. The next section describes how to determine whether or not this is the case.

Discovering our criterion

At first glance, it might seem that conditions 1–3 should be enough for a digraph of various arrow-types to determine a group up to isomorphism. To see that these conditions are not sufficient, consider the digraph in Figure 4 with arrow-types identified as a and b. Note that we have chosen the labels b and $baab$ for the vertices arrived at when beginning at e and traversing the paths described by these labels. We will often choose to identify a vertex v with its label $\lambda(v)$.

It is easy to check that Γ satisfies conditions 1–3. However, while a and $baab$ are two sequences of arrow-types that both end at vertex $baab$ when beginning at vertex e, these same sequences end at different vertices when beginning at vertex b. Thus,

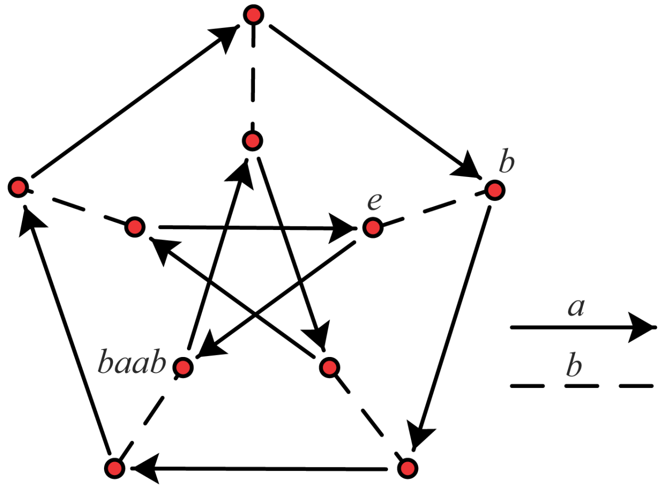

Figure 4 $ea = ebaab$, but $ba \neq bbaab$.

condition 4 is not satisfied, and we conclude that Γ is not a Cayley digraph for any group.

Let's see how much we can glean from this example.

Consider the function $\pi : V(\Gamma) \to V(\Gamma)$ from the vertex set of Γ to itself given by $\pi(v) = b\lambda(v)$. In essence, this function changes the starting vertex from e to b, and identifies the new vertex arrived at by starting at b and traversing the arrow-types given by $\lambda(v)$ for each vertex v. In our example, $(e, baab)$ is a directed edge of arrow-type a. However, $(\pi(e), \pi(baab))$ has arrow-type a^{-1}. This is an essential observation—if Γ *had* satisfied condition 4, then π would necessarily have preserved arrow-types. Let's see why.

Suppose Γ' is a Cayley digraph for some group, and let λ be a labeling for Γ'. Let $k \in V(\Gamma')$ and let $\pi_k : V(\Gamma') \to V(\Gamma')$ be given by $\pi_k(z) = k\lambda(z)$ for each vertex z. Observe that if v_1 and v_2 are two vertices of Γ', then

$$\pi_k(v_1) = \pi_k(v_2) \iff kv_1 = kv_2 \iff v_1 = v_2$$

since $V(\Gamma')$ is a group with respect to the natural operation described earlier. In other words, if Γ' is a Cayley digraph for some group then the functions π_k are *permutations*. We will now show that these permutations preserve arrow-types. Let v be a vertex of Γ', and let w be the vertex of Γ' such that edge (v, w) has arrow-type a. Regardless of the label assigned to w, it can also be expressed as va. (Recall that (v, va) must exist by condition 2.) Note that $\lambda(w) = \lambda(va)$ is a path of arrow-types that begins at e and ends at vertex $w = va$. Similarly, $\lambda(v)a$ is a path of arrow-types that begins at e and ends at va (with v as the penultimate vertex visited). By condition 4, it follows that $k\lambda(va)$ is equal to $k\lambda(v)a$; that is, $\pi_k(va) = \pi_k(v)a$. Since $(\pi_k(v), \pi_k(v)a)$ is clearly an edge of arrow-type a, it follows that $(\pi_k(v), \pi_k(va))$ is that same edge of arrow-type a. So (v, w) and $(\pi_k(v), \pi_k(w))$ are both edges of arrow-type a. This shows that *for each vertex k, the permutation π_k preserves arrow-types* (i.e., each π_k is a graph automorphism).

Fortunately, it turns out that this condition is also *sufficient* (in the presence of conditions 1–3) to recognize Cayley digraphs. To see this, let Γ' be a digraph of various arrow-types that satisfies conditions 1–3, and let λ denote any labeling of Γ'. Assume that π_k (defined as above) is an arrow-type preserving permutation for each vertex k. We will show that Γ' satisfies condition 4. Let v and w be vertices of Γ', and let

$a_1 \cdots a_n$ and $b_1 \cdots b_m$ be sequences of arrow-types for which

$$va_1 \cdots a_n = vb_1 \cdots b_m = w.$$

Since π_v preserves arrow-types, we have by condition 2 that

$$\begin{aligned}
va_1a_2 \cdots a_{n-1}a_n &= ((\cdots((va_1)a_2)\cdots)a_{n-1})a_n \\
&= ((\cdots(\pi_v(a_1)a_2)\cdots)a_{n-1})a_n \\
&= ((\cdots(\pi_v(a_1a_2))\cdots)a_{n-1})a_n \\
&\vdots \\
&= \pi_v(a_1a_2\cdots a_n).
\end{aligned}$$

Similarly, $vb_1 \cdots b_m = \pi_v(b_1 \cdots b_m)$. So, since π_v is a permutation, it follows that $a_1 \cdots a_n$ and $b_1 \cdots b_m$ represent the same vertex of Γ'. Thus, for any other vertex v' we must have that $\pi_{v'}(a_1 \cdots a_n) = \pi_{v'}(b_1 \cdots b_m)$. And, since $\pi_{v'}$ preserves arrow-types, we have that $v'a_1 \cdots a_n = v'b_1 \cdots b_m$ by reversing the argument given above (substituting $\pi_{v'}$ for π_v).

Let's summarize what we've just discovered. Let Γ be a digraph of various arrow-types that satisfies conditions 1–3, and let λ be any labeling of Γ. For each k in the vertex set of Γ, define $\pi_k : V(\Gamma) \to V(\Gamma)$ by $\pi_k(v) = k\lambda(v)$. Then

Theorem 2. *Γ is a Cayley digraph of some group if and only if π_k is an arrow-type preserving permutation for each k in the vertex set of Γ.*

Theorem 2 permits us to determine whether or not a graph is Cayley in a finite number of steps, but the task of testing π_k for every vertex k could be formidable. Fortunately, we can do better. Suppose we know only that π_a is a permutation that preserves arrow-types *for each arrow-type a* of Γ. If a_1, a_2, and a_3 are all arrow-types it follows that

$$a_1(a_2a_3) = \pi_{a_1}(a_2a_3) = (\pi_{a_1}(a_2))a_3 = (a_1a_2)a_3.$$

This argument can be extended by induction to show that any sequence of arrow-types can be regrouped without affecting the vertex that the given sequence represents. Therefore, for any vertices v and w, and any arrow-type a, we have that

$$\begin{aligned}
\pi_v(wa) &= v(wa) = \lambda(v)(\lambda(w)a) \\
&= (\lambda(v)\lambda(w))a = (v\lambda(w))a = (\pi_v(w))a.
\end{aligned}$$

In other words, *if π_a is an arrow-type preserving permutation for every arrow-type a then π_v preserves arrow-types for every vertex v*. Moreover, each π_v is a permutation since for any vertices w and z with $\pi_v(w) = \pi_v(z)$ we have that

$$\begin{aligned}
\pi_v(w) = \pi_v(z) &\iff v\lambda(w) = v\lambda(z) \\
&\iff v_1 \cdots v_n\lambda(w) = v_1 \cdots v_n\lambda(z) \\
&\iff v_1(v_2 \cdots v_n\lambda(w)) = v_1(v_2 \cdots v_n\lambda(z)) \\
&\iff \pi_{v_1}(v_2 \cdots v_n\lambda(w)) = \pi_{v_1}(v_2 \cdots v_n\lambda(z)) \\
&\iff v_2 \cdots v_n\lambda(w) = v_2 \cdots v_n\lambda(z) \\
&\vdots
\end{aligned}$$

$$\iff \lambda(w) = \lambda(z)$$

$$\iff w = z$$

where $v_1, \ldots, v_n$ are arrow-types with $\lambda(v) = v_1 \cdots v_n$.

In light of the condition discovered above, this observation provides the result we have been seeking: Let Γ be a digraph of various arrow-types that satisfies conditions 1–3, and let λ be any labeling of Γ. For each arrow-type a of Γ, define $\pi_a : V(\Gamma) \to V(\Gamma)$ by $\pi_a(v) = a\lambda(v)$. Then

Theorem 3 (Condition 4′). *Γ is a Cayley digraph of some group if and only if π_a is an arrow-type preserving permutation for each arrow-type a.*

The following procedure uses this criterion to determine whether a finite digraph Γ of various arrow-types represents a Cayley graph of some group.

1. Verify that Γ satisfies conditions 1–3.
2. Choose a vertex to represent the identity element e. Label the other vertices v of Γ with sequences of arrow-types that describe a path from e to v.
3. Draw another picture of Γ. Call it Γ'. Choose an arrow-type a of Γ and identify the vertex of Γ represented by a. Let e' denote the corresponding vertex of Γ'. Beginning at e' and using the labels chosen for the vertices of Γ, label all of the other vertices of Γ'. (If this process cannot be completed then STOP—π_a is not a permutation and therefore Γ is not a Cayley digraph of any group.)
4. For each edge (v, w) of Γ, determine whether there is an edge of Γ' of the same arrow-type and with the same vertex labels. (If this fails for any edge then STOP—π_a does not preserve arrow-types and therefore Γ is not a Cayley digraph of any group.)
5. Repeat the last 2 steps for each remaining arrow-type of Γ. If these steps can be completed for each arrow-type before stopping, Γ is a Cayley digraph of some group.

Applying our criterion

Let us use these steps to reexamine the graph that we looked at earlier (reproduced in the first diagram of Figure 5). When vertex e is repositioned at vertex b, vertices b and $baab$ are consequently repositioned as shown. In the new graph, $(baab, e)$ is an edge of arrow-type a. However, in the original graph $(baab, e)$ is an edge of arrow type

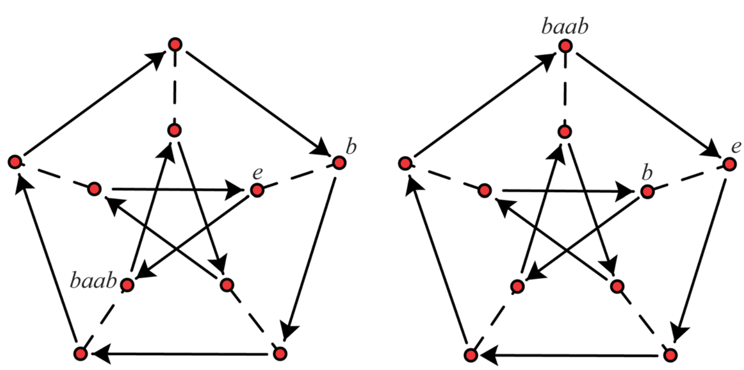

Figure 5 Not a Cayley digraph—the arrow-type of edge $(baab, e)$ is not preserved.

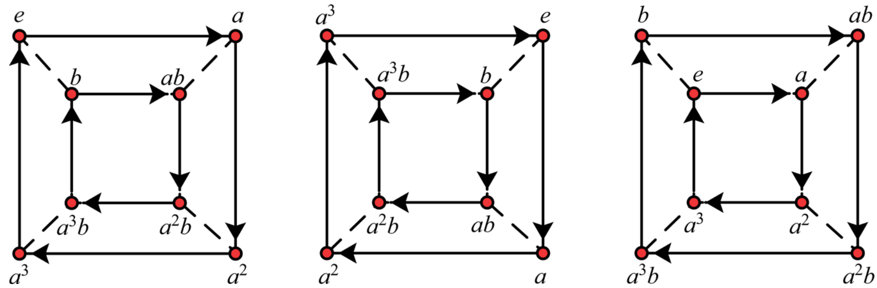

Figure 6 Using our criterion to recognize a Cayley digraph.

a^{-1}. Step 4 of our procedure instructs us to stop as we have determined that π_b is not arrow-type preserving. The graph fails to be a Cayley digraph of any group.

As promised, condition 4′ also provides a classroom-friendly means to determine that a finite digraph of various arrow-types represents a group—something that condition 4 did not permit.

Let Γ denote the first digraph labeled in Figure 6 with edge types a and b. The other digraphs are identical to Γ but with vertices labeled as described in the procedure above. We refer to them as Γ' and Γ'' respectively. It is quick to check that Γ satisfies conditions 1–3 and that the edges of each of Γ' and Γ'' have the same vertex labels and arrow-types as do the corresponding edges of Γ. Therefore, Γ is a Cayley graph for some group G. (In this example, G is isomorphic to $\mathbf{Z}_4 \times \mathbf{Z}_2$.)

Algebraic graphs

Let Γ denote a digraph with edges of various arrow types, and let λ be a labeling of Γ. As we have seen, conditions 1–3 are insufficient to recognize whether Γ represents a group. However, satisfying these conditions *is* sufficient for $v * w = v\lambda(w)$ to define an operation on the vertex set $V(\Gamma)$. As such, $V(\Gamma)$ together with $*$ can be regarded as an algebraic structure that we will call a Γ-*algebra*. For this reason, we use the term *algebraic graph* to refer to any digraph with edges of various arrow-types that satisfies conditions 1–3. To the best of our knowledge, the notions of Γ-algebra and algebraic graph are new with this article.

We already know a few things about Γ-algebras. By our definition of a labeling, these algebraic structures have an identity element e (the vertex represented by the empty label). Also, it is easy to show that the operation $*$ is not necessarily associative (see below); however, $*$ *is* associative provided that the functions $\pi_a : V(\Gamma) \to V(\Gamma)$ (defined by $\pi_a(v) = av$) are permutations that preserve arrow-types for each arrow-type a of Γ. (In fact, we have seen that this condition is actually enough to determine that the Γ-algebra is a group.) It is immediate that if $\lambda(v) = v_1 \cdots v_n$, then the vertex arrived at by traversing this path backwards $(v_n^{-1} \cdots v_1^{-1})$ from e is a left inverse of v. It takes a bit more thought to recognize that $v_n^{-1} \cdots v_1^{-1}$ is *not* necessarily a right inverse of v. This is because the label assigned to this vertex might not describe a path of arrow-types that ends at e when beginning at v. All of these assertions become more evident by considering the labeled algebraic graph in Figure 7.

The operation that is induced by this labeling is described in the table that follows. At first glance, this operation table appears to define a group. Clearly e serves as an identity element, and each label appears exactly once in each row and column of the table. However, upon closer inspection we see that $*$ is not an associative operation

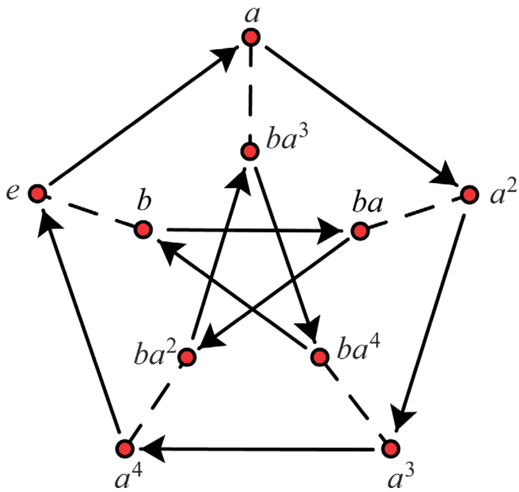

Figure 7 A labeling λ of an algebraic graph Γ.

Table 1: Operation table for the Γ-algebra induced by λ

	e	a	a^2	a^3	a^4	b	ba	ba^2	ba^3	ba^4
e	e	a	a^2	a^3	a^4	b	ba	ba^2	ba^3	ba^4
a	a	a^2	a^3	a^4	e	ba^3	ba^4	b	ba	ba^2
a^2	a^2	a^3	a^4	e	a	ba	ba^2	ba^3	ba^4	b
a^3	a^3	a^4	e	a	a^2	ba^4	b	ba	ba^2	ba^3
a^4	a^4	e	a	a^2	a^3	ba^2	ba^3	ba^4	b	ba
b	b	ba	ba^2	ba^3	ba^4	e	a	a^2	a^3	a^4
ba	ba	ba^2	ba^3	ba^4	b	a^2	a^3	a^4	e	a
ba^2	ba^2	ba^3	ba^4	b	ba	a^4	e	a	a^2	a^3
ba^3	ba^3	ba^4	b	ba	ba^2	a	a^2	a^3	a^4	e
ba^4	ba^4	b	ba	ba^2	ba^3	a^3	a^4	e	a	a^2

since (for example)

$$(ba^2 * a^2) * ba = a^4$$

while

$$ba^2 * (a^2 * ba) = a.$$

Moreover, while every element has both a left inverse element and a right inverse element, these inverses are not always identical. (Note that the left and right inverses of ba^3 are ba and ba^4, respectively.)

It might be surprising to realize that, in general, Γ-algebras are dependent upon the choice of labeling of a given algebraic graph. Note that if v is the vertex labeled ba in Figure 7 then v^2 is the vertex labeled a^3. However, if we instead use a^2b as the label for v then v^2 is equal to a. Since $\sigma(v^2) = [\sigma(v)]^2$ for any isomorphism σ, this observation shows that even the identity permutation might fail to be an isomorphism between the Γ-algebras induced on a given algebraic graph by different labelings. This immediately suggests two very natural questions:

Question 2. *Are Cayley graphs the only algebraic graphs whose associated Γ-algebras are independent of the choice of labeling?*

Question 3. *How many distinct Γ-algebras exist (up to isomorphism) for a given algebraic graph?*

Concluding remarks

Cayley graphs are perhaps the most efficient and visually pleasing means for representing the algebraic structure of a group. In an abstract algebra course, these representations can be used as an intuitive introduction to the concept of isomorphism—two groups are isomorphic provided that they admit identical Cayley graphs. Toward this end, it is instructive to draw as many algebraic graphs as possible on a set of, say, eight vertices. Having done so, one might then examine properties (e.g., commutativity, order of elements, etc.) of each of these potential groups to get a sense of whether any pair of them might be isomorphic. I have found this to be a fun and accessible way to introduce the notions of isomorphism and properties of groups near the very start of the course. The techniques presented in this article provide an accessible means by which to identify which of these proposed groups are *really* groups. Moreover, by considering algebraic graphs and Γ-algebras in an abstract algebra course, algebraic structures that are more general than groups can be introduced in a very natural way.

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Summary. A criterion is developed that decides in a finite number of steps whether a finite digraph is a Cayley digraph of some group. In the process, a class of algebraic structure more general than groups are considered that arise naturally from digraphs that are "almost" Cayley.

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A Generalization of the One-Seventh Ellipse

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There is a curious mathematical phenomenon called the *One-Seventh Ellipse* [1, 3]. Take the digits from the decimal expansion of $1/7$, namely 142857, and form six points in the plane: $(1, 4)$, $(4, 2)$, $(2, 8)$, $(8, 5)$, $(5, 7)$, and $(7, 1)$. The surprising fact is that these six points lie on an ellipse. Moreover, if we take consecutive digits from the decimal expansion two at a time, the six points $(14, 28)$, $(42, 85)$, $(28, 57)$, $(85, 71)$, $(57, 14)$, and $(71, 42)$ also lie on an ellipse. Figure 1 displays both ellipses. This note explains and generalizes these observations.

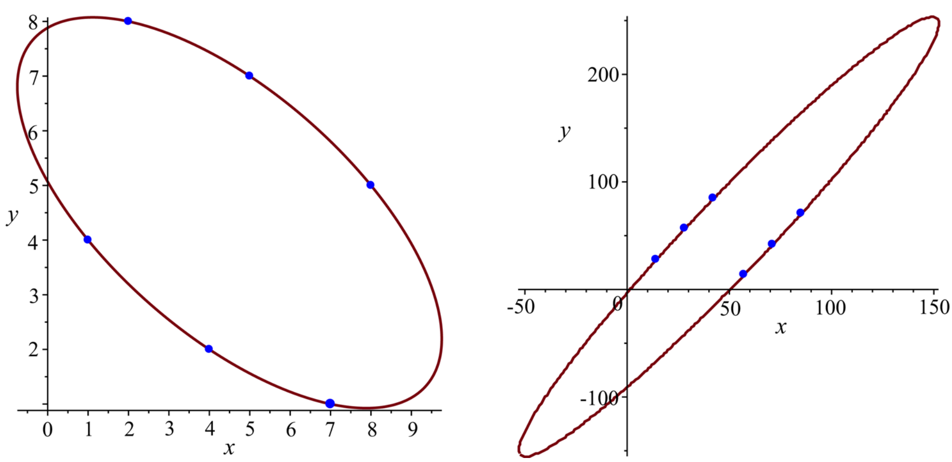

Figure 1 First and second ellipse.

It is natural to think that this phenomenon is related to number theory. However, the tools we need come from geometry. From the set of points used in the one-seventh ellipse, notice that $1 + 8 = 4 + 5 = 2 + 7$, and also that $14 + 85 = 42 + 57 = 28 + 71$. More generally, we find that the sequence of six numbers

$$z_1 = a, \quad z_2 = b, \quad z_3 = c,$$
$$z_4 = S - a, \quad z_5 = S - b, \quad z_6 = S - c$$

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MSC: 51A99

can be used to generate sets of six points in the plane (assuming $z_1, \ldots, z_6$ are distinct) that exhibit a beautiful structure. For each $n = 1, \ldots, 6$, define P_n as the set of six ordered pairs (z_i, z_{i+n}), $i = 1, \ldots, 6$, where wrap-around is used if necessary. When we refer to a conic section, we do not limit ourselves to the traditional curves obtained by slicing a cone with a plane, but, rather, the locus of a quadratic polynomial in two variables. This includes ellipses, hyperbolas, a line, or a pair of parallel lines.

Theorem 1. *Suppose that a, b, c, $S - a$, $S - b$, $S - c$ are six distinct real numbers. Then for each $n = 1, \ldots, 6$, we have*

1. *The six points in P_n lie on a unique conic section. The center of each conic is located at $(S/2, S/2)$. For $n = 3$, the conic is the line $x + y = S$, while for $n = 6$, the conic is the line $x = y$.*
2. *If $n \neq n'$ are in $\{1, 2, 4, 5\}$, then $P_{n'}$ can be obtained by applying a reflection to P_n. The same reflection connects the conics associated with each sets of points.*

To prove the theorem, we will need some results from geometry. For context, we first state Pascal's hexagrammum mysticum theorem (see Figure 2).

Theorem 2 (Pascal)**.** *If a hexagon is inscribed in a conic, then the three points of intersection of the lines which contain opposite sides of the hexagon are collinear.*

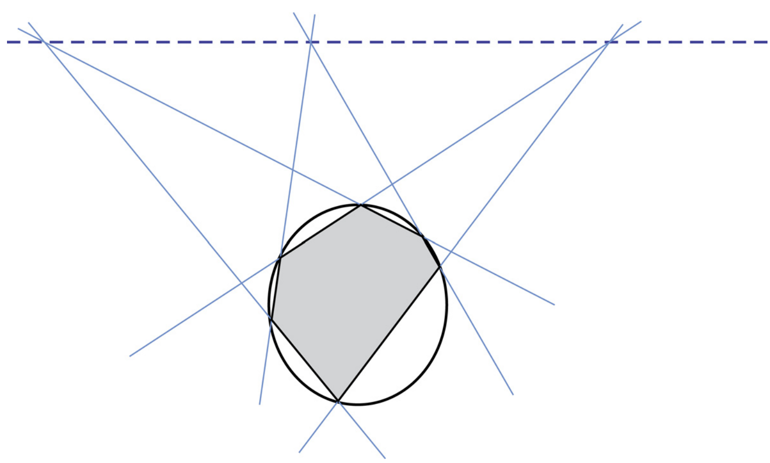

Figure 2 Pascal's hexagrammum mysticum theorem.

Pascal's theorem has an interesting converse.

Theorem 3 (Braikenridge–Maclaurin)**.** *If three lines meet three other lines in nine points, and three of these points lie on a line, then the remaining six points lie on a conic.*

Of interest here is that these theorems extend to the case of parallel lines when one adds a point at infinity. We are really working in the projective plane $\mathbb{R}P^2$. These results are described in Traves [**2**]. Using Theorem 3, we can now prove Theorem 1.

Proof. In proving part (1), the cases $n = 3$ and $n = 6$ are trivial. Consider the set P_1. Form two sets of three lines:

$$\begin{aligned} S_1 = \{ \ & \text{line through } (a, b) \text{ and } (b, c), \\ & \text{line through } (c, S - a) \text{ and } (S - a, S - b), \\ & \text{line through } (S - b, S - c) \text{ and } (S - c, a). \} \end{aligned}$$

and

$$S_2 = \{ \text{ line through } (b, c) \text{ and } (c, S-a), \\ \text{line through } (S-a, S-b) \text{ and } (S-b, S-c), \\ \text{line through } (S-c, a) \text{ and } (a, b). \}$$

By computing their slopes, one sees that each line in S_1 is parallel to a line in S_2. This implies that three of the nine points of intersection of S_1 and S_2 are the point at infinity, so these three points are collinear in $\mathbb{R}P^2$. By the Braikenridge–Maclaurin theorem, the remaining six points of intersection, namely the points in P_1, lie on a conic. A similar approach applies to the sets P_2, P_4, and P_5.

To prove part (2), we use reflections across the following lines: $y = x$, $x + y = S$, $x = S/2$ and $y = S/2$. One can show that P_j and P_k, where $j, k \in \{1, 2, 4, 5\}$ and $j \neq k$, are related by one or two of these reflections. For example,

$$P_1 = \{(a, b), (b, c), (c, S-a), (S-a, S-b), (S-b, S-c), (S-c, a)\}$$

can be reflected about the line $x = S/2$ to produce

$$P_4 = \{(S-a, b), (S-b, c), (S-c, S-a), (a, S-b), (b, S-a), (c, a)\}.$$

We leave it to the reader to check the other connections.

To show that each conic section is also a result of the same reflections, observe that reflecting a conic section results in a conic section. Since six points determine at most one conic section, the set of reflections that take P_j to P_k will also take the conic through P_j to the conic through P_k.

■

Examples are straightforward to produce. To form an ellipse, take the original sequence $\{1, 4, 2, 8, 5, 7\}$. To form a hyperbola, use the sequence $\{5, 4, 3, 7, 8, 9\}$. To form a pair of parallel lines, take the sequence $\{1, 2, 4, 8, 7, 5\}$. Figures 3–5 show these three cases.

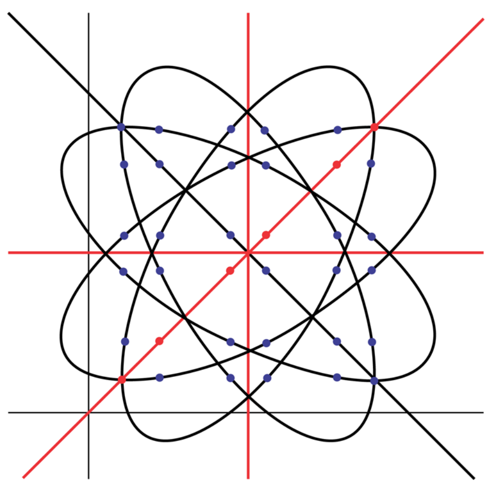

Figure 3 Ellipses.

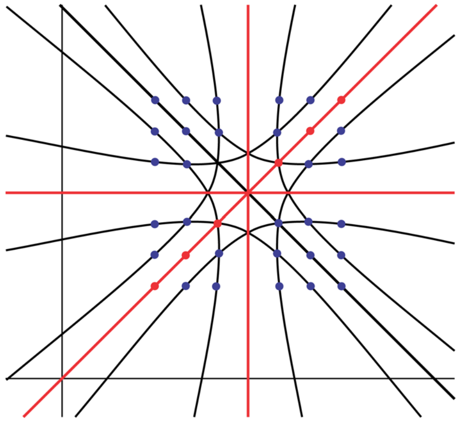

Figure 4 Hyperbolas.

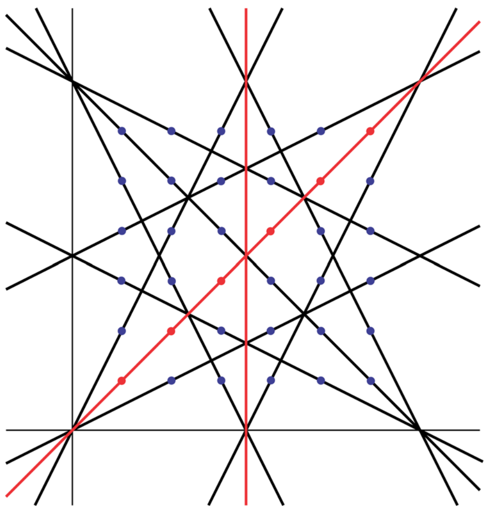

Figure 5 Parallel lines.

While Theorem 1 applies to any choice of $a, b, c \in \mathbb{R}$, suppose we try to connect this back to the original $1/7$ observation and limit the parameters to

$$a, b, c, S-a, S-b, S-c \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

For which $x \in \mathbb{Q}$ will this produce a similar phenomenon? Of course, this will produce only a finite number of possibilities. Writing the decimal expansion of x as an infinite series, we find

$$\begin{aligned} x &= \frac{a}{10} + \frac{b}{10^2} + \frac{c}{10^3} + \frac{S-a}{10^4} + \frac{S-b}{10^5} + \frac{S-c}{10^6} + \cdots \\ &= \left(\frac{a}{10} + \frac{b}{10^2} + \frac{c}{10^3} + \frac{S-a}{10^4} + \frac{S-b}{10^5} + \frac{S-c}{10^6}\right)\left(1 + \frac{1}{10^6} + \frac{1}{10^{12}} + \cdots\right) \\ &= \left[\frac{999}{1000}\left(\frac{a}{10} + \frac{b}{10^2} + \frac{c}{10^3}\right) + \frac{S}{1000}\left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3}\right)\right]\left(\frac{1}{1 - \frac{1}{10^6}}\right) \end{aligned}$$

$$= [999(100a + 10b + c) + 111S]\,\frac{1}{10^6 - 1}$$
$$= \frac{100a + 10b + c}{1001} + \frac{S}{9009}.$$

For example, letting $a = 1$, $b = 2$, $c = 5$, and $S = 9$ produces the number $x = 18/143$. One easily determines that the associated conics are ellipses.

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Summary. Using the digits from the decimal expansion of 1/7, namely 142857, and forming six points in the plane—(1, 4), (4, 2), (2, 8), (8, 5), (5, 7), (7, 1)—one finds, surprisingly, that these six points lie on an ellipse. This note explains and generalizes this observation.

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Elementary Proofs of the Trigonometric Inequalities of Huygens, Cusa, and Wilker

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Students and instructors in first-year calculus will recall that the limit of $(\sin x)/x$ is 1 as x approaches 0, which is used to find the derivatives of the trigonometric functions. The usual procedure is to establish the double inequality

$$1 > \frac{\sin x}{x} > \cos x$$

for $0 < |x| < \frac{\pi}{2}$ and appeal to the squeeze theorem. In this note, we discuss sharper bounds for $(\sin x)/x$, several of which are consequences of inequalities that have been known for several centuries. We also show that those bounds are familiar means of the bounds in the double inequality above, i.e., means of 1 and $\cos x$.

The inequality of Huygens in our title is

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 3 \tag{1}$$

for $0 < |x| < \frac{\pi}{2}$ (see, e.g., Baricz and Sándor [**1**], Larson [**3**, Problem 7.4.12], or Neuman and Sándor[**5**, Sect. 1.3]). Although Sándor [**6**] refers to (1) as "the famous Huyhens's inequality," it does not appear to be very well known. In his classic 1654 work *De Circuli Magnitudine Inventa* (On Finding the Magnitude of the Circle) the Dutch astronomer and physicist Christiaan Huygens (1629–1695) used (1) to approximate the value of π, thus the name for the inequality. It is sometimes referred to as *Snell's inequality* for the Dutch astronomer and mathematician Willebrord Snellius (1580–1626), having appeared in the form

$$2\sin\frac{x}{3} + \tan\frac{x}{3} > x$$

for $0 < x < \frac{3\pi}{2}$ in his 1621 work *Cyclometricus* (Measuring the Circle).

We begin by presenting an elementary proof of (1) using calculus after first establishing a simple lemma.

Lemma 1. *For* $0 < t < \frac{\pi}{2}$*, we have that*

$$\cos t > 1 - \frac{t^2}{2} \quad \textit{and} \quad \sec^2 t > 1 + t^2.$$

Proof. Comparing the lengths of the chord AB and the arc AB on the left in Figure 1 yields

$$\sqrt{(\cos t - 1)^2 + \sin^2 t} < t,$$

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MSC: Primary 26D05, Secondary 33B10; 26E60

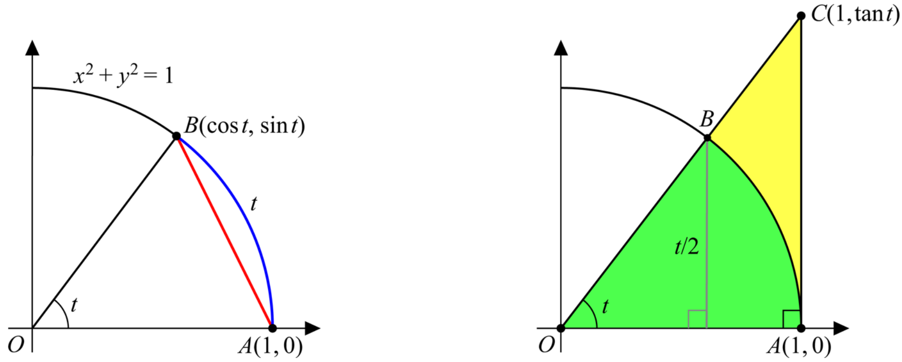

Figure 1 Inequalities for $\cos t$ and $\sec^2 t$.

so that $2 - 2\cos t < t^2$ and hence $\cos t > 1 - \frac{t^2}{2}$. On the right we have area($\triangle AOC$) > area(sectorAOB), so that $\frac{1}{2}\tan t > \frac{1}{2}t$ and hence

$$\sec^2 t = 1 + \tan^2 t > 1 + t^2.$$

■

Theorem 1 (Huygens's inequality)**.** *For* $0 < |x| < \frac{\pi}{2}$, *we have that*

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 3.$$

Proof. Since $(\sin x)/x$ and $(\tan x)/x$ are even functions, it suffices to show that $2\sin x + \tan x > 3x$ for $0 < x < \frac{\pi}{2}$. Integrating the inequalities from Lemma 1 yields

$$\sin x = \int_0^x \cos t \, dt > \int_0^x \left(1 - \frac{t^2}{2}\right) dt = x - \frac{x^3}{6}, \tag{2}$$

$$\tan x = \int_0^x \sec^2 t \, dt > \int_0^x (1 + t^2) \, dt = x + \frac{x^3}{3},$$

and thus

$$2\sin x + \tan x > \left(2x - \frac{x^3}{3}\right) + \left(x + \frac{x^3}{3}\right) = 3x.$$

■

The inequality in the following corollary appeared in the January 1989 issue of the *American Mathematical Monthly* in a problem (E 3306) proposed by J. B. Wilker, hence its name. See Baricz and Sándor [1] and Neuman and Sándor [5] for details.

Corollary 1 (Wilker's inequality)**.** *For* $0 < |x| < \frac{\pi}{2}$, *we have that*

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2.$$

Proof. Since $(\sin x)/x$ and $(\tan x)/x$ are even functions, it suffices to establish the inequality for $0 < x < \frac{\pi}{2}$. Since $\sin x > x - \frac{x^3}{6}$ for $0 < x < \frac{\pi}{2}$ we have

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > \left(1 - \frac{x^2}{6}\right)^2 + \left(1 + \frac{x^2}{3}\right) = 2 + \frac{x^4}{36} > 2.$$

■

Additional inequalities for both circular and hyperbolic functions can be similarly established. We present two simple but useful ones. The one in the next theorem (see Neuman and Sándor [**5**, equation (1.1)]) also appeared in Huygens's *De Circuli Magnitudine Inventa*, and is attributed to the German philosopher Nicolas of Cusa (1401–1464). It is known in the literature as Cusa's inequality (or the Cusa–Huygens inequality) and, like Huygens's inequality, does not appear to be well-known. The inequality in Theorem 3, an inequality of Adamović and Mitrinović that is the circular analogue of a hyperbolic inequality of Lazarević, appears in Bhayo and Sándor [**2**], Larson [**3**, Problem 7.5.4], and Mitrinović [**4**, Inequality 3.4.18]. The proofs make repeated use of (2) to generate inequalities involving partial sums of the power series for the sine and cosine.

Theorem 2 (Cusa's inequality)**.** *For* $0 < |x| < \frac{\pi}{2}$*, we have that*

$$\frac{\sin x}{x} < \frac{2+\cos x}{3}.$$

Proof. Since the terms in the inequality are even functions, we need only consider $0 < x < \frac{\pi}{2}$. From (2), we have

$$1-\cos x = \int_0^x \sin t \, dt > \int_0^x \left(t - \frac{t^3}{6}\right) dt = \frac{x^2}{2} - \frac{x^4}{24},$$

so that

$$\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Then

$$\sin x = \int_0^x \cos t \, dt < \int_0^x \left(1 - \frac{t^2}{2} + \frac{t^4}{24}\right) dt = x - \frac{x^3}{6} + \frac{x^5}{120},$$

and hence

$$3\frac{\sin x}{x} < 3 - \frac{x^2}{2} + \frac{x^4}{40}.$$

Integrating again yields

$$1-\cos x = \int_0^x \sin t \, dt < \int_0^x \left(t - \frac{t^3}{6} + \frac{t^5}{120}\right) dt = \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720},$$

hence

$$3 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} < 2 + \cos x.$$

So it suffices to show that

$$3 - \frac{x^2}{2} + \frac{x^4}{40} < 3 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720},$$

or equivalently that

$$\frac{x^2}{720} < \frac{1}{24} - \frac{1}{40}.$$

That is, $x^2 < 12$, which is true, since $0 < x < \frac{\pi}{2}$. ■

Theorem 3. *For* $0 < |x| < \frac{\pi}{2}$, *we have that*

$$\frac{\sin x}{x} > \cos^{1/3} x.$$

Proof. As before, we need only consider $0 < x < \frac{\pi}{2}$. From (2), we have

$$\left(\frac{\sin x}{x}\right)^3 > \left(1 - \frac{x^2}{6}\right)^3 = 1 - \frac{x^2}{2} + \frac{x^4}{12} - \frac{x^6}{216},$$

and from the proof of the previous theorem we have that

$$1 - \frac{x^2}{2} + \frac{x^4}{24} > \cos x.$$

So it suffices to show that

$$1 - \frac{x^2}{2} + \frac{x^4}{12} - \frac{x^6}{216} > 1 - \frac{x^2}{2} + \frac{x^4}{24},$$

or equivalently

$$\frac{1}{12} - \frac{1}{24} > \frac{x^2}{216}.$$

That is, $9 > x^2$, which is true since $0 < x < \frac{\pi}{2}$. ■

The inequalities in Theorems 2 and 3 combine to yield the double inequality

$$\frac{2 + \cos x}{3} > \frac{\sin x}{x} > \cos^{1/3} x,$$

for $0 < |x| < \frac{\pi}{2}$. This is inequality (1.1) in Neuman and Sándor [5]. Note that these bounds for $(\sin x)/x$ are the arithmetic mean and the geometric mean of the three numbers 1, 1, and $\cos x$ (or equivalently, weighted means of 1 and $\cos x$ with weights 2/3 and 1/3, respectively). Furthermore, Huygens's inequality (1) is equivalent to

$$\frac{\sin x}{x} > \frac{3\cos x}{2\cos x + 1} = \frac{3}{1 + 1 + (1/\cos x)}.$$

That is, $(\sin x)/x$ is greater than the harmonic mean of 1, 1, and $\cos x$. Since $(\sin x)/x$ and the three means are all even functions of x, well known inequalities between the means yield

$$\frac{2 + \cos x}{3} > \frac{\sin x}{x} > \cos^{1/3} x > \frac{3\cos x}{2\cos x + 1}. \tag{3}$$

for $0 < |x| < \frac{\pi}{2}$.

In Figure 2, we present graphs of $(\sin x)/x$ and the three means for x in $(-\pi/2, \pi/2)$.

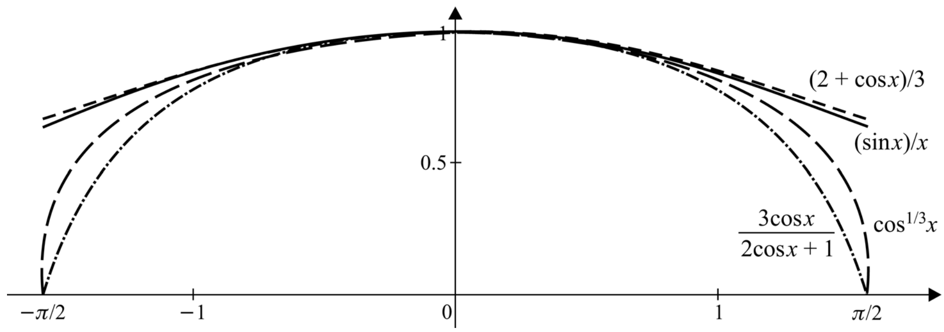

Figure 2 Graphs of the functions in (3).

Example 1. Figure 2 suggests that when $x = \pi/n$ for large n, Cusa'a inequality in the form $x > 3\sin x/(2+\cos x)$ may provide an excellent approximation to π, i.e.,

$$\pi \approx \frac{3n\sin(\pi/n)}{2+\cos(\pi/n)}.$$

To evaluate $\sin(\pi/n)$ and $\cos(\pi/n)$ without using π we begin with $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$, then employ the half angle formulas for the sine and cosine to compute $3n\sin(\pi/n)/(2+\cos(\pi/n))$ when n is a power of 2. We arrange the work in the following table, computing each row (after the $n = 2$ row) from the previous row. The

TABLE 1: Using Cusa's inequality to approximate π.

n	$\sin(\pi/n)$	$\cos(\pi/n)$	$\frac{3n\sin(\pi/n)}{2+\cos(\pi/n)}$
2	1.0	0.0	3.0
4	0.7071067812	0.7071067812	3.1344464996
8	0.3826834324	0.9238795325	3.1411698993
16	0.1950903220	0.9807852804	3.1415665926
32	0.0980171403	0.9951847267	3.1415910302
64	0.0490676743	0.9987954562	3.1415925526
128	0.0245412285	0.9996988187	3.1415926475
256	0.0122715383	0.9999247018	3.1415926532

last two approximations agree (rounded to 8 places), yielding $\pi \approx 3.14159265$.

The inequality in Theorem 3 and the arithmetic mean-geometric mean inequality yield one-line proofs of Theorem 1 and Corollary 1. Since proving Huygens's inequality (1) is equivalent to showing that

$$\frac{1}{3}\left(\frac{\sin x}{x} + \frac{\sin x}{x} + \frac{\sin x}{x\cos x}\right) > 1,$$

we have

$$\frac{1}{3}\left(\frac{\sin x}{x} + \frac{\sin x}{x} + \frac{\sin x}{x\cos x}\right) > \left(\frac{\sin x}{x}\cdot\frac{\sin x}{x}\cdot\frac{\sin x}{x\cos x}\right)^{1/3} = \frac{\sin x}{x\cos^{1/3} x} > 1.$$

Similarly, for Wilker's inequality in Corollary 1 we have

$$\frac{1}{2}\left[\left(\frac{\sin x}{x}\right)^2+\frac{\sin x}{x\cos x}\right]>\left[\left(\frac{\sin x}{x}\right)^2\cdot\frac{\sin x}{x\cos x}\right]^{1/2}=\left(\frac{\sin x}{x\cos^{1/3}x}\right)^{3/2}>1.$$

We conclude with a brief discussion of some hyperbolic versions of the above inequalities. The hyperbolic version of Huygens's inequality in Neuman and Sándor [5, Corollary 2.3] is

$$2\frac{\sinh x}{x}+\frac{\tanh x}{x}>3.$$

for $|x|>0$. The proof is similar to that for the circular version. In the left-hand image in Figure 3, we first recall that a hyperbolic sector AOB, like the circular sector, has area $t/2$, and derive two inequalities in the next lemma.

Lemma 2. *For $t>0$, we have*

$$\cosh t>1+\frac{t^2}{2}\quad \textit{and}\quad \operatorname{sech}^2 t>1-t^2.$$

Proof. In the right-hand image in Figure 3, we have area($\triangle AOB$) > area(sectorAOB), so that $\frac{1}{2}\sinh t>\frac{1}{2}t$. Integration yields

$$\cosh t-1=\int_0^t \sinh u\,du>\int_0^t u\,du=\frac{t^2}{2}.$$

We also have area(sectorAOB) > area($\triangle AOC$), so that $\frac{1}{2}t>\frac{1}{2}\tanh t$. Hence,

$$\operatorname{sech}^2 t=1-\tanh^2 t>1-t^2.$$

∎

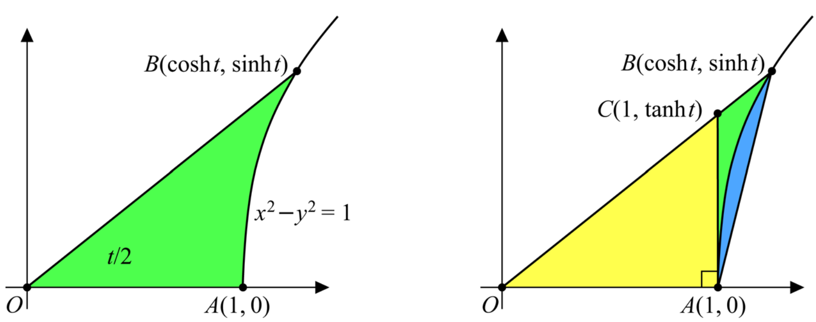

Figure 3 Inequalities for $\cosh t$ and $\operatorname{sech}^2 t$.

Theorem 4. *For $|x|>0$, we have that*

$$2\frac{\sinh x}{x}+\frac{\tanh x}{x}>3.$$

Proof. As in the proof of Theorem 1, we need only show that

$$2\sinh x + \tanh x > 3x,$$

for $x > 0$. From Lemma 2 we have

$$\sinh x = \int_0^x \cosh t\, dt > \int_0^x \left(1 + \frac{t^2}{2}\right) = x + \frac{x^3}{6},$$
$$\tanh x = \int_0^x \operatorname{sech}^2 t\, dt > \int_0^x (1 - t^2)\, dt = x - \frac{x^3}{3},$$

and thus

$$2\sinh x + \tanh x > \left(2x + \frac{x^3}{3}\right) + \left(x - \frac{x^3}{3}\right) = 3x.$$

■

Corollary 2. *For* $|x| > 0$, *we have*

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2.$$

The proof is analogous to that of Corollary 1 and is omitted. Hyperbolic versions of the inequalities in Theorems 2 and 3 exist (see Neuman and Sándor[**5**] and Sándor [**6**]) and lead to the following string of inequalities analogous to those in (3):

$$\frac{2 + \cosh x}{3} > \frac{\sinh x}{x} > \cosh^{1/3} x > \frac{3\cosh x}{2\cosh x + 1}, \quad \text{for } |x| > 0. \tag{4}$$

See Figure 4 for graphs of the functions in (4).

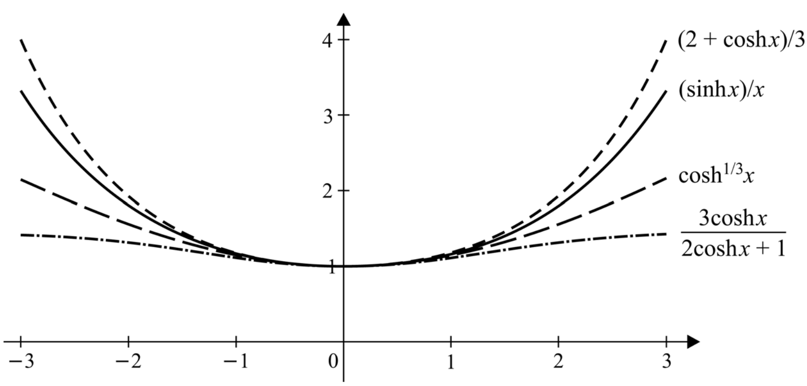

Figure 4 Graphs of functions in (4).

Example 2. Noting that

$$\sinh(\ln x) = \frac{(x^2 - 1)}{2x} \quad \text{and} \quad \cosh(\ln x) = \frac{(x^2 + 1)}{2x}$$

suggests that deleting the term $\cosh^{1/3} x$ in (4) and rearranging the remaining inequalities will yield rational functions as bounds on $\ln x$, i.e.,

$$\frac{3\sinh(\ln x)}{2+\cosh(\ln x)} = \frac{3(x^2-1)}{x^2+4x+1} < \ln x < \frac{(x+1)(x^3-1)}{3x(x^2+1)} = \frac{(2\cosh(\ln x)+1)\sinh(\ln x)}{3\cosh(\ln x)},$$

for $x > 1$ (the inequalities are reversed for $0 < x < 1$). For example, when $x = 1.1$ we have

$$0.095310136 < \ln 1.1 < 0.095310572.$$

The publication dates of items 1, 2, 5, and 6 in the bibliography indicate that interest in trigonometric inequalities from the 15th and 17th centuries continues in the 21st century, with current work yielding generalizations, extensions, and applications of those inequalities. The paper by Bhayo and Sándor [2] is an excellent survey of some recent results.

Acknowledgments The author wishes to thank three anonymous reviewers and the editor for many helpful suggestions on an earlier draft of this article.

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Summary. We present elementary proofs of the inequalities of Huygens, Cusa, and Wilker, using visual proofs of several simpler inequalities and integral calculus.

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Graph Cycles and Olympiad Problems

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We present four simple lemmas concerning the existence of cycles in graphs, and show how the lemmas can be applied to solve problems that appeared in mathematical competitions. Despite their simplicity, these tools can be used to tackle surprisingly varied problems.

We begin with some basics of graph theory. A *graph* G consists of a set of *vertices* V and a set of *edges* E. The graph is said to be *finite* if both V and E are finite.

In a *directed graph*, each edge is directed from a vertex v to a vertex w, where possibly $v = w$. We allow multiple edges to be directed between the same pair of vertices. A *directed cycle* is a sequence of vertices $v_1, v_2, \ldots, v_n$, for some $n \geq 1$, such that there is a directed edge $v_i \to v_{i+1}$ for all $i = 1, 2, \ldots, n-1$ and a directed edge $v_n \to v_1$. (If $n = 1$, we require a directed edge $v_1 \to v_1$.) The *out-degree* of a vertex is the number of edges directed from that vertex, and the *in-degree* of a vertex is the number of edges directed into that vertex.

Similarly, in an *undirected graph*, each edge connects two vertices v and w, where possibly $v = w$. We allow multiple edges to connect the same pair of vertices. An *undirected cycle* is a sequence of vertices $v_1, v_2, \ldots, v_n$, for some $n \geq 1$, such that there is an undirected edge between v_i and v_{i+1} for all $i = 1, 2, \ldots, n-1$ and an undirected edge between v_n and v_1. (If $n = 1$, we require an undirected edge between v_1 and itself.) The *degree* of a vertex is the number of edges adjacent to that vertex. (An edge between a vertex and itself is counted twice in the degree of that vertex.)

We are now ready to state the lemmas. First, we consider directed graphs.

Lemma 1. *Let G be a finite directed graph. If every vertex of G has out-degree at least* 1*, then G has a directed cycle.*

Proof. Suppose that every vertex of G has out-degree at least 1. Let v_1 be an arbitrary vertex of G. Given vertex v_i, define v_{i+1} as any vertex for which there exists an edge $v_i \to v_{i+1}$. Since the graph is finite, there exist two indices $j < k$ such that $v_j = v_k$. Then the path $v_j \to v_{j+1} \to \cdots \to v_k$ forms a directed cycle. ■

Lemma 2. *Let G be a finite directed graph. If every vertex of G has out-degree* 1 *and in-degree* 1*, then G is a disjoint union of directed cycles.*

Proof. Suppose that every vertex of G has out-degree 1 and in-degree 1. Let v_1 be an arbitrary vertex of G. Given vertex v_i, define v_{i+1} as the vertex for which there exists an edge $v_i \to v_{i+1}$. Since the graph is finite, there exist two indices $j < k$ such that $v_j = v_k$. Consider the first vertex v_k such that there exists some $j < k$ for which $v_j = v_k$. If $j > 1$, then v_j has in-degree at least 2, contradicting the assumption. Hence $j = 1$, and the path $v_1 \to v_2 \to \cdots \to v_{k-1} \to v_1$ forms a directed cycle. We can remove it and apply mathematical induction on the remaining graph, which consists of strictly fewer vertices. ■

We also have equivalent results on undirected graphs. The proofs are similar and are left as exercises for the reader.

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Lemma 3. *Let G be a finite undirected graph. If every vertex of G has degree at least 2, then G has an undirected cycle.*

Lemma 4. *Let G be a finite undirected graph. If every vertex of G has degree 2, then G is a disjoint union of undirected cycles.*

The key in applying these lemmas is to identify the appropriate graph G for each problem. We illustrate this through some problems given in mathematical competitions.

Our first problem involves a simple fact from matching theory, an extremely rich subject on its own (see Roth and Sotomayor [1], for example). In the language of matching theory, it states that in a "one-sided matching," for every assignment in the "core" of the game, at least one person receives his or her top choice. The problem appeared in the Turkish Mathematical Olympiad in 1998, and is closely related to the Top Trading Cycle Algorithm [2].

Problem 1. *There are n people who need to be assigned to n houses. Each person ranks the houses in some order, with no ties. After the assignment is made, it is observed that every other assignment would assign at least one person to a house that the person ranks lower than the house in the given assignment. Prove that at least one person receives his or her top choice in the given assignment.*

Proof. Assume that the ith person has been assigned to the ith house. Construct a directed graph G with vertices $1, 2, \ldots, n$. For each person i, add an edge from i to i's favorite house. Every vertex of G has out-degree 1, so by Lemma 1, G contains a directed cycle. If the cycle consists of a single vertex, we have found a person who receives his or her top choice in the given assignment. Otherwise, we can let the people in the cycle trade their houses along the cycle, making all of them happier and thus contradicting the condition of the problem. ■

Our next problem appeared in the Iranian Mathematical Olympiad in 1998.

Problem 2. *An $n \times n$ table is filled with the numbers $-1, 0, 1$ in such a way that every row and column of the table contains exactly one -1 and one 1. (An example is shown below.) Prove that one can permute the rows and columns so that in the resulting table each number is the negative of the number in the same position in the original table.*

Proof. Consider a directed graph G whose vertices correspond to the rows of the table. For each column, add an edge to G pointing from the row in which the column has a 1 to the row in which the column has a -1. For example, for this table:

$$\begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

the graph G consists of the edges $4 \to 2 \to 1 \to 3 \to 4$. Every vertex of G has out-degree 1 and in-degree 1, so by Lemma 2, G is a disjoint union of directed cycles. Consider any directed cycle

$$s_1 \to s_2 \to \cdots \to s_k \to s_1$$

in G, and assume that row s_i has a 1 in column t_i. (For the cycle $4 \to 2 \to 1 \to 3 \to 4$ in the above table, we have $t_1 = 1$, $t_2 = 2$, $t_3 = 4$, and $t_4 = 3$.) We switch rows s_2

and s_k, rows s_3 and s_{k-1}, ..., and rows $s_{\lceil k/2\rceil}$ and $s_{\lfloor k/2\rfloor+2}$. Note that this has the effect of reversing the cycle. We then switch columns t_1 and t_k, columns t_2 and t_{k-1}, ..., and columns $t_{\lfloor k/2\rfloor}$ and $t_{\lceil k/2\rceil+1}$. It can be verified that after this procedure, all 1's in these rows and columns are replaced by -1's, and vice versa. Hence, after we perform the procedure on all cycles in G, we obtain the desired table. ∎

Our next problem appeared in the Russian Mathematical Olympiad in 2005.

Problem 3. *A conference has* 100 *participants from* 50 *countries, two from each country. The participants sit at a round table. Prove that one may partition them into two groups in such a way that no two participants from the same country are in the same group, and no three consecutive participants in the circle are in the same group.*

Proof. Number the participants in the round table $1, 2, \ldots, 100$ in the order in which they are seated, and pair them up as $\{1, 2\}, \{3, 4\}, \ldots, \{99, 100\}$. Construct an undirected graph G with the vertices corresponding to the participants. For each participant, add an edge to her pair and an edge to the other participant from the same country. Every vertex of G has degree 2, so by Lemma 4, G is a disjoint union of undirected cycles. Since the edges in a cycle necessarily alternate between the "pair" type and the "same country" type, each cycle must be of even length. Hence, we can partition the participants in each cycle into two groups so that no two participants in the same group are connected by an edge. It follows that no two participants from the same country are in the same group, and no three consecutive participants in the circle are in the same group. ∎

Our next problem appeared in the Peru Team Selection Test for the International Mathematical Olympiad in 2006.

Problem 4. *A table with* 2^n *rows and* n *columns is filled with* 1 *and* -1 *in such a way that the rows of the table constitute all possible sequences of length* n *that can be formed with* 1 *and* -1. *An arbitrary subset of numbers is then replaced by* 0*s. (An example is shown below.) Prove that one can choose a nonempty subset of rows of the table so that within the chosen rows, the sum of the numbers in each column is 0.*

$$\begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{pmatrix}$$

Proof. We refer to a row in the table before modification as an *original row*, and one after modification as a *modified row*. We define a *binary row* to be a row with n entries consisting of 0 and 1. Let

$$f : \{0, 1\}^n \to \{-1, 1\}^n$$

be a function such that for each binary row b, $f(b)$ is the original row for which the entry of $f(b)$ is 1 if the corresponding entry of b is 0, and the entry of $f(b)$ is -1 if the corresponding entry of b is 1. (For example, $f(0, 0) = (1, 1)$ and $f(1, 0) = (-1, 1)$.) Let

$$g : \{0, 1\}^n \to \{-1, 0, 1\}^n$$

be such that $g(b)$ is the modified row that coincides with $f(b)$ before we made any change to the table, but possibly has had some entries replaced by 0. (So in the above table, $g(0, 0) = (1, 1)$ and $g(1, 0) = (-1, 0)$.) One can check that $b + g(b)$ is again a binary row, where addition is done entrywise. Construct a directed graph G with the vertices corresponding to all 2^n binary rows. For each binary row b, add an edge to the binary row $b + g(b)$. Every vertex of G has out-degree 1, so by Lemma 1, G contains a directed cycle $b_1, b_2, \ldots, b_k$. Hence

$$b_1 + g(b_1) + g(b_2) + \cdots + g(b_k) = b_1,$$

which implies that $g(b_1), g(b_2), \ldots, g(b_k)$ form a desired subset of rows. ■

Our final problem appeared on the shortlist of the International Mathematical Olympiad in 2017.

Problem 5. *Determine all integers $n \geq 2$ with the following property: for any integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by n, there exists an index $1 \leq i \leq n$ such that none of the numbers*

$$a_i, a_i + a_{i+1}, \ldots, a_i + a_{i+1} + \cdots + a_{i+n-1}$$

is divisible by n. (We let $a_i = a_{i-n}$ when $i > n$.)

Proof. First, if n is composite, say $n = ab$ for integers $a, b \geq 2$, then we can take

$$(a_1, a_2, \ldots, a_{n-1}, a_n) = (a, a, \ldots, a, 0)$$

to show that the property does not hold.

Next, let n be a prime number, and assume for the sake of contradiction that the property does not hold. Construct a directed graph G with vertices $1, 2, \ldots, n$. For each a_i, there exists $1 \leq j \leq n - 1$ such that

$$a_i + a_{i+1} + \cdots + a_{i+j-1}$$

is divisible by n. Add a directed edge from i to $i + j$ in G, where we take the vertex indices modulo n. Every vertex of G has out-degree 1, so by Lemma 1, G contains a directed cycle $v_1, v_2, \ldots, v_k$. This means that the sum

$$(a_{v_1} + a_{v_1+1} + \cdots + a_{v_2-1}) + (a_{v_2} + a_{v_2+1} + \cdots + a_{v_3-1}) + \cdots + (a_{v_k} + a_{v_k+1} + \cdots + a_{v_1-1})$$

is divisible by n. This sum contains each a_i an equal number of times, and this number can be at most $n - 1$. Hence, we have that $r(a_1 + a_2 + \cdots + a_n)$ is divisible by n for some $1 \leq r \leq n - 1$. However, this is impossible since n is prime and neither r nor $a_1 + a_2 + \cdots + a_n$ is divisible by n. ■

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Summary. We show how certain basic results about cycles in directed and undirected graphs can be used to solve some clever problems that appeared in major mathematics competitions.

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Why Does a Prime p Divide a Fermat Number?

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Pierre de Fermat (1601–1665) spent his entire working life in Toulouse, first as commissioner of requests (1631–1648) and then as King's councillor (1648–1665) in the local parliament. He was a scrupulously honest, even-tempered, just man who lived very quietly [**1**, pp. 293–294], [**2**, pp. 57–58]. His recreation was mathematics, to which he made many significant contributions, especially in number theory. Although Fermat published only a few papers, several of his methods and results bear his name more than 350 years after his death: Fermat's method of descent, Fermat's little theorem, Fermat's factorization method, the Fermat–Girard theorem on primes as sums of two squares, and even the famous Fermat's last theorem, the proof of which by Andrew Wiles has contributed so much to modern mathematics.

During the course of Fermat's correspondence with Bernard Frénicle (1605–1670), Fermat discussed the possible divisors of the natural numbers $2^k + 1$ $(k = 1, 2, 3, \ldots)$ and expressed the conviction that if k is a power of 2 then $2^k + 1$ is a prime. Indeed

$$2^{2^0} + 1 = 3, \quad 2^{2^1} + 1 = 5, \quad 2^{2^2} + 1 = 17,$$

$$2^{2^3} + 1 = 257, \quad \text{and} \quad 2^{2^4} + 1 = 65537,$$

are all primes. It is not known if there are infinitely many primes in the sequence $\{2^{2^m} + 1 \mid m = 0, 1, 2, \ldots\}$, although Hardy and Wright [**9**, p. 14] give a heuristic argument suggesting that there are only a finite number. Recently, Boklan and Conway [**4**] have provided compelling evidence that in fact the only numbers of the form $2^{2^m} + 1$ $(m = 0, 1, 2, \ldots)$ which are prime are the five known to Fermat! In honor of Fermat, the numbers $F_m := 2^{2^m} + 1$ $(m = 0, 1, 2, \ldots)$ are called Fermat numbers and those which are prime are called Fermat primes. If F_m is a composite number, then the primes dividing F_m are called Fermat prime divisors. Since two different Fermat numbers have no common factors [**9**, p. 14], a Fermat prime divisor divides one and only one Fermat number.

In 1732, Euler showed that $F_5 = 2^{2^5} + 1 = 4294967297$ is divisible by 641 and so is not a prime, disproving Fermat's assertion. Euler's elegant argument that 641 divides F_5 without doing the actual division [**9**, p. 14] can be easily extended to give the complete factorization of F_5 without doing any division except by 5 (if you know what you are doing!), namely

$$\begin{aligned} F_5 = 2^{32} + 1 = 2^{18} \cdot 2^{14} + 1 = 262144 \cdot 2^{14} + 1 = 261735 \cdot 2^{14} + 409 \cdot 2^{14} + 1 \\ = 261735 \cdot 2^{14} + 52352 \cdot 2^7 + 1 = 5 \cdot 52347 \cdot 2^{14} + (5 + 52347) \cdot 2^7 + 1 \\ = (5 \cdot 2^7 + 1)(52347 \cdot 2^7 + 1) = 641 \cdot 6700417. \end{aligned}$$

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The numbers 641 and 6700417 are both primes, so they are Fermat prime divisors. Later, Clausen, in an unpublished letter to Gauss in 1855, mentioned that he had factored F_6 (see Biermann [3, p. 185]). Landry [11] (at the age of 82) in 1880 found that 274177 divides F_6. Analogously to the almost division-free calculation for F_5, we have (allowing just one division by 1071)

$$\begin{aligned}
F_6 &= 2^{64} + 1 = 2^{48} \cdot 2^{16} + 1 \\
&= 281474976710656 \cdot 2^{16} + 1 \\
&= 281473950092895 \cdot 2^{16} + 1026617761 \cdot 2^{16} + 1 \\
&= 1071 \cdot 262814145745 \cdot 2^{16} + 262814146816 \cdot 2^{8} + 1 \\
&= 1071 \cdot 262814145745 \cdot 2^{16} + (1071 + 262814145745) \cdot 2^{8} + 1 \\
&= (1071 \cdot 2^{8} + 1)(262814145745 \cdot 2^{8} + 1) \\
&= 274177 \cdot 67280421310721.
\end{aligned}$$

Both 274177 and 67280421310721 are primes, so they are Fermat prime divisors. A worldwide parallel computing project is currently underway searching for Fermat prime divisors. For information about this project, the reader should consult www.fermatsearch.org. Wilfrid Keller [10] keeps a detailed account of all known Fermat prime divisors and their discoverers. As of July 27, 2020 the number of known Fermat prime divisors stood at 353. The latest known one is $171369935 \cdot 2^{11077} + 1$, which has 3343 digits and divides F_{11075}.

There are a number of methods used to find Fermat prime divisors, see, for example, Morrison and Brillhart [13] (continued fraction method), Brent and Pollard [6] (Pollard rho algorithm), Lenstra et al. [12] (number field sieve), and Brent [5] (elliptic curve method). It is not our purpose here to describe these factoring methods but rather to understand what it is in the “internal structure” of a prime that makes it a Fermat prime divisor. What do Fermat prime divisors such as 641 and 274177 have in common?

If p is a Fermat prime divisor, then p divides $2^{2^m} + 1$ for some positive integer m. Therefore, $p \neq 2$. Since

$$\left(2^{2^{m-1}}\right)^2 \equiv -1 \pmod{p}$$

the prime p must satisfy $p \equiv 1 \pmod 4$. By the Fermat–Girard theorem, it follows that p is the sum of two squares. This is the “internal structure” that we use to say whether or not p is a Fermat prime divisor.

An important result of Lucas, which has its origin in the work of Legendre and Euler [14, pp. 101–102], gives a larger power of 2 than 2^2 in $p - 1$ for a Fermat prime divisor p.

Theorem 1 (Lucas’ criterion)**.** *If the prime p divides F_m, where $m \geq 2$, then $p = 2^{m+2}k + 1$ for some positive integer k.*

We note that the positive integer k in Lucas’ criterion may be odd or even. Lucas’ criterion only gives candidates for possible divisors of Fermat numbers. However, it does eliminate certain primes as Fermat prime divisors. For example, 37 is never a Fermat prime divisor. To see this, suppose to the contrary that 37 divides F_m, for some non-negative integer m. Clearly, 37 does not divide $F_0 = 3$, and $F_1 = 5$, implying that we may suppose $m \geq 2$. By Lucas’ criterion, we have $37 = 2^{m+2}k + 1$ for some positive integer k. Hence, 2^{m+2} divides 36, contradicting $m \geq 2$. At the end of this article, by way of contrast, we use our theorem (Theorem 2) to prove that 37 is not a Fermat prime divisor.

We now introduce the notation we need to state our theorem. Let p be a prime satisfying $p \equiv 1 \pmod 4$. We define a to be the largest positive integer such that 2^a divides $p-1$ (so that $a \geq 2$), and b to be the positive odd integer $(p-1)/2^a$, so that $p = 2^a b + 1$. The integers a and b are uniquely determined by p. Clearly, b is not divisible by p. Since $p \equiv 1 \pmod 4$, by the Fermat–Girard theorem there are positive integers R and S uniquely determined by p such that

$$p = R^2 + S^2, \;\; R \equiv 1 \,(\mathrm{mod}\ 2), \;\; S \equiv 0 \,(\mathrm{mod}\ 2).$$

Brillhart [7] has given a very fast algorithm for determining R and S. His algorithm starts by finding the unique integer x $(0 < x < p/2)$ satisfying $x^2 \equiv -1 \pmod p$, and then applies the Euclidean algorithm to p/x, stopping at the first remainder less than $p^{1/2}$. Clearly, R and S are coprime, since p is a prime and neither R nor S is divisible by p. We answer the question "Why does a prime p divide a Fermat number?" in the following theorem in terms of a divisibility condition involving a, b, R, and S.

Theorem 2. *Let $p \equiv 1 \pmod 4$ be a prime which is not a Fermat number. Then p is a Fermat prime divisor if and only if there exists a positive integer m and $\epsilon = \pm 1$ such that p divides M, where*

$$M := \begin{cases} 2^{2^{m-1}-a} R - \epsilon b S & \text{if } a \leq 2^{m-1}, \\ R - \epsilon 2^{a-2^{m-1}} b S & \text{if } a > 2^{m-1}. \end{cases} \tag{1}$$

Before giving the proof of Theorem 2, we show how the theorem reduces the problem of determining the smallest prime factor p (necessarily $p \equiv 1 \pmod 4$) of a composite Fermat number F_m to that of the divisibility of a much smaller integer by p. Such a prime factor p satisfies $p \leq F_m^{1/2}$ and so R and S, which are both smaller than $p^{1/2}$, satisfy $R, S \leq F_m^{1/4}$. By Theorem 2, p divides the integer M defined in (1). If $a \leq 2^{m-1}$ we have

$$\begin{aligned} |M| &\leq (2^{2^{m-1}-a} + b) F_m^{1/4} \\ &= \frac{(2^{2^{m-1}} + 2^a b)}{2^a} F_m^{1/4} < \frac{(F_m^{1/2} + p)}{4} F_m^{1/4} \leq \frac{1}{2} F_m^{3/4}, \end{aligned}$$

and if $a > 2^{m-1}$ we have

$$|M| \leq (1 + 2^a b) F_m^{1/4} = p F_m^{1/4} \leq F_m^{3/4}.$$

Thus the integer M to be tested for divisibility by p is considerably smaller in absolute value than F_m.

In each row of Table 1, the first column contains the number of digits of the specified Fermat number F_m, the second column the smallest prime divisor p of F_m, the third and fourth columns the values of R and S such that $p = R^2 + S^2$, and the fifth column the number of digits in the integer M defined in (1). For the values of m in the table, the number of digits in M is approximately half the number of digits in F_m.

Proof. Suppose first that there is a positive integer m and $\epsilon = \pm 1$ such that p divides M. Then

$$x := \frac{2^{2^{m-1}} R - \epsilon 2^a b S}{p} = \begin{cases} 2^a M/p & \text{if } a \leq 2^{m-1}, \\ 2^{2^{m-1}} M/p & \text{if } a > 2^{m-1}, \end{cases}$$

TABLE 1: Comparison between the number of digits of F_m and the number of digits of M for various m, where M is defined in (1) and p is the smallest prime dividing F_m

F_m	$p = 2^a b + 1$	R	S	M
F_6 (20 digits)	274177	89	516	10 digits
F_7 (39 digits)	59649589127497217	208648999	126945596	25 digits
F_8 (78 digits)	1238926361552897	24246559	25515304	43 digits
F_9 (155 digits)	2424833	127	1552	75 digits
F_{10} (309 digits)	45592577	2279	6356	154 digits
F_{11} (617 digits)	319489	167	540	307 digits
F_{12} (1234 digits)	114689	217	260	615 digits
F_{13} (2467 digits)	2710954639361	755681	1462840	1235 digits
F_{15} (9865 digits)	1214251009	34815	1472	4931 digits
F_{16} (19729 digits)	825753601	19425	21176	9863 digits
F_{18} (78914 digits)	13631489	25	3692	39452 digits
F_{19} (157827 digits)	70525124609	243047	107020	78913 digits
F_{21} (631306 digits)	4485296422913	607847	2028748	315652 digits
F_{23} (2525223 digits)	167772161	12455	3556	1262608 digits

is an integer. From

$$R(2^{2^{m-1}} S + \epsilon 2^a b R) = (Sx + \epsilon 2^a b) p,$$

we deduce that

$$y := (2^{2^{m-1}} S + \epsilon 2^a b R)/p$$

is an integer. Thus,

$$\begin{aligned} p^2(x^2 + y^2) &= (2^{2^{m-1}} R - \epsilon 2^a b S)^2 + (2^{2^{m-1}} S + \epsilon 2^a b R)^2 \\ &= (R^2 + S^2)(2^{2^m} + 2^{2a} b^2) = p(2^{2^m} + 2^{2a} b^2), \end{aligned}$$

which implies that

$$p(x^2 + y^2) = 2^{2^m} + 2^{2a} b^2.$$

Hence,

$$\begin{aligned} p(x^2 + y^2 - (2^a b - 1)) &= 2^{2^m} + 2^{2a} b^2 - (2^a b + 1)(2^a b - 1) \\ &= 2^{2^m} + 1 = F_m, \end{aligned}$$

so that p divides F_m.

We now show that $p \neq F_m$, so that F_m is composite and p is a Fermat prime divisor. If $p = F_m$, then

$$x^2 + y^2 - (2^a b - 1) = 1.$$

That is,

$$x^2 + y^2 = 2^a b,$$

so

$$2^{2^m} + 2^{2a} b^2 = p(x^2 + y^2) = 2^a b p.$$

Thus,

$$2^{2^m-a}+2^ab^2=bp.$$

Hence, $2^{2^m-a}+2^ab^2$ is an odd positive integer. Since $a\ge 2$, we must have $2^{2^m-a}=1$, $a=2^m$ and $b=1$. Therefore, $p=2^{2^m}+1$, contradicting that p is not a Fermat number.

Now we suppose that $p=2^ab+1$ is a Fermat prime divisor. Then $p\mid F_m$ for some positive integer m. Hence, as mentioned earlier, p is a prime congruent to $1 \pmod 4$, and so there are unique positive integers R and S with $p=R^2+S^2$, $\gcd(R,S)=1$, $R\equiv 1 \pmod 2$ and $S\equiv 0 \pmod 2$. Since

$$\begin{aligned} 2^{2^m} &= F_m-1\equiv -1 \pmod p,\\ S^2 &= p-R^2\equiv -R^2 \pmod p, \quad \text{and}\\ (2^ab)^2 &= (p-1)^2\equiv 1 \pmod p, \end{aligned}$$

we see that

$$2^{2^m}R^2-S^2(2^ab)^2\equiv 0 \pmod p.$$

Then, since

$$2^{2^m}R^2-2^{2a}b^2S^2=(2^{2^{m-1}}R-2^abS)(2^{2^{m-1}}R+2^abS),$$

we see that p divides $2^{2^{m-1}}R-\epsilon 2^abS$ for $\epsilon=1$ or $\epsilon=-1$. Now $2^{2^{m-1}}R-\epsilon 2^abS$ is 2^aM if $a\le 2^{m-1}$ and is $2^{2^{m-1}}M$ if $a>2^{m-1}$ so p divides M. ■

We note that if p divides M, then p divides F_m, and conversely, if p divides F_m, then p divides M. We also note that if p divides M, then it does so for exactly one of the two values of ϵ, since p does not divide R. It is also the case that the integer M in Theorem 2 can be replaced by the integer M' defined as

$$M' := \begin{cases} 2^{2^{m-1}-a}S+\epsilon bR & \text{if } a\le 2^{m-1},\\ S+\epsilon 2^{a-2^{m-1}}bR & \text{if } a>2^{m-1}, \end{cases}$$

since $M^2+M'^2$ is a multiple of p, and thus p divides M if and only if p divides M'. This observation may be advantageous when $|M'|<|M|$.

If N is an arbitrary positive integer with $N\equiv 1 \pmod 4$, then there may not exist positive integers R and S such that $N=R^2+S^2,\ \gcd(R,S)=1$.

An example is $N=9$. On the other hand there may exist more than one such pair (R,S), for example, $N=65=1^2+8^2=7^2+4^2$. If, however, there is one such pair for which $N=2^ab+1$ divides M, where the integer M is defined in (1), then

$$\begin{aligned} \gcd(R,N)&=\gcd(R,R^2+S^2)=\gcd(R,S^2)=1,\\ \gcd(S,N)&=\gcd(S,R^2+S^2)=\gcd(S,R^2)=1. \end{aligned}$$

In a manner similar to the proof of Theorem 2, we deduce that N divides F_m. Thus, we can apply Theorem 2 to such N, whether N is a prime or not.

We illustrate Theorem 2 by applying it to some of the divisors of F_5, F_6, F_7, F_8, and F_9.

- **Euler 1732:** 641 divides F_5 since

$$641 = 2^7 \cdot 5 + 1 = 25^2 + 4^2, \quad 2^9 \cdot 25 + 5 \cdot 4 = 20 \cdot 641.$$

- **Euler 1732:** 6700417 divides F_5 since

$$6700417 = 2^7 \cdot 52347 + 1 = 409^2 + 2556^2$$

$$2^9 \cdot 409 + 52347 \cdot 2556 = 20 \cdot 6700417.$$

- **Clausen, 1855; Landry, 1880 [11]:** 274177 divides F_6 since

$$274177 = 2^8 \cdot 1071 + 1 = 89^2 + 516^2,$$

$$2^{24} \cdot 89 - 1071 \cdot 516 = 5444 \cdot 274177.$$

- **Morrison and Brillhart 1970 [13]:** Let

$$N = 59649589127497217, \quad b = 116503103764643,$$
$$R = 208648999, \quad S = 126945596, \quad T = 125777612.$$

  Then N divides F_7 since

$$N = 2^9 b + 1 = R^2 + S^2, \quad 2^{55} R - bS = TN.$$

- **Brent and Pollard 1980 [6]:** Let

$$N = 1238926361552897, \quad b = 604944512477,$$
$$R = 24246559, \quad S = 25515304,$$
$$T = 3251727264065752656690508920.$$

  Then N divides F_8 since

$$N = 2^{11} b + 1 = R^2 + S^2, \quad 2^{117} R - bS = TN.$$

- **Western 1903 [8]:** 2424833 divides F_9 since

$$2424833 = 2^{16} \cdot 37 + 1 = 127^2 + 1552^2,$$

$$2^{240} \cdot 127 - 37 \cdot 1552 = 2424833 \cdot T,$$

  where T has 68 digits.

We conclude by using Theorem 2 to show that the prime 37 is not a Fermat prime divisor. Here, $a = 2$, $b = 9$, $R = 1$, and $S = 6$. Suppose that 37 is a Fermat prime divisor. Then, by Theorem 2, there exists a positive integer m and $\varepsilon = \pm 1$ such that

$$2^{2^{m-1}} - \varepsilon 216 \equiv 0 \pmod{37}.$$

Thus

$$2^{2^{m-1}} \equiv (\varepsilon 6)^3 \pmod{37}.$$

Define integers h and k by

$$h = [2^{m-1}/3],\ k = 2^{m-1} - 3[2^{m-1}/3],$$

so that $2^{m-1} = 3h + k$ and $k = 1$ or 2. Then, since $2 \cdot 19 \equiv 1 \pmod{37}$, we have

$$2^k = 2^{2^{m-1}-3h} \equiv 2^{2^{m-1}} 19^{3h} \equiv (\varepsilon 6 \cdot 19^h)^3 \pmod{37},$$

so that 2 or 4 is a cube modulo 37. But this is a contradiction as the 12 cubes modulo 37 are

$$1 \equiv 1^3,\ 6 \equiv 31^3,\ 8 \equiv 2^3,\ 10 \equiv 34^3,\ 11 \equiv 25^3,\ 14 \equiv 13^3,\ 23 \equiv 32^3,$$
$$26 \equiv 16^3,\ 27 \equiv 3^3,\ 29 \equiv 17^3,\ 31 \equiv 8^3,\ 36 \equiv 27^3,$$

proving that 37 is not a Fermat prime divisor.

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Summary. A prime dividing a composite Fermat number is called a Fermat prime divisor. Such a prime p must be congruent to 1 modulo 4, and so, by the Fermat–Girard theorem, there exists integers R and S such that $p = R^2 + S^2$. We derive a necessary and sufficient condition for p to be a Fermat prime divisor in terms of the integers R and S.

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True Grit in Real Analysis

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"When I use a word," Humpty Dumpty said, in a rather scornful tone, "it means just what I choose it to mean—neither more nor less." "The question is," said Alice, "whether you can make words mean so many different things." "The question is," said Humpty Dumpty, "which is to be master—that's all."—Lewis Carroll, *Through the Looking Glass* [7]

The axiomatic or Euclidean description of mathematics is unmatched for clarity and precision. It is the ideal toward which all mathematical theories aspire, and one could assert that no theory is truly mathematical if it cannot be so rendered. At the same time, one of the great obstacles faced by undergraduate mathematics majors is understanding and appreciating this style of presentation. Nowhere is this more apparent than in Real Analysis, a course that is usually taught in the purest Euclidean format and which students find confusing, unmotivated, and daunting.

Lakatos's insight

The definitions of real analysis lie at the core of student difficulties. Like Humpty Dumpty in *Through the Looking Glass*, mathematicians use words to mean whatever they want, and each word means neither more nor less than its definition. Barbara Edwards [9] has given an account of the difficulties that students encounter when faced with this use of definition. She has found two necessary conditions for students to be able to work successfully with mathematical definitions. The first condition is that students must realize that mathematicians are not using definitions as they are usually encountered, as descriptions of entities that already exist. For mathematicians, definitions are prescriptive. The second condition is that students must understand the ideas behind and the reasons for each particular definition, a context in which to place it.

Imre Lakatos has a lot to say about mathematical definitions. In his *Proofs and Refutations* [12], he shows how precise definitions emerge from a process in which patterns are observed, theorems and their proofs are discovered, and then counter-examples are produced, forcing a re-examination of underlying assumptions and definitions. This leads to new proofs and new counter-examples in a cycle of proofs and refutations that eventually produces the highly refined definitions and proofs that are codified in textbooks. While Lakatos's principal illustration comes from the history of Euler's formula for the relationship of the numbers of vertices, edges, and faces in a polyhedron, he uses an appendix to illustrate how this process was central to the development of analysis.

As Lakatos realized, appreciation of this dynamic is essential for understanding modern mathematics, and passage directly to Euclidean rigor is pedagogically indefensible:

> The Euclidean method can, in certain problem situations, have deleterious effects on the development of mathematics. Most of these problem situations occur

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> in growing mathematical theories, where growing concepts are the vehicles of progress, where the most exciting developments come from exploring the boundary regions of concepts, from stretching them, and from differentiating formerly undifferentiated concepts. In these growing theories intuition is inexperienced, it stumbles and errs. There is no theory which has not passed through such a period of growth; moreover, this period is the most exciting from the historical point of view. These periods cannot be properly understood without understanding the method of proofs and refutations, without adopting a fallibilist approach.
>
> This is why Euclid has been the evil genius particularly for the history of mathematics and for the teaching of mathematics, both on the introductory and the creative levels. [**12**, p. 140]

I became convinced that senior mathematics students must be exposed to this process, and that in this exposure lies the opportunity to explore the nature of mathematical definition and to wrestle with the problems that led to the modern definitions of such concepts as continuity, uniform convergence, integrability, and measure. Lakatos's account of Cauchy and the concept of uniform convergence became one of the pivotal sections of *A Radical Approach to Real Analysis* [**4**].

Cauchy's error

Cauchy's *Cours d'analyse* [**8**] of 1821 is actually a pre-calculus textbook written for college freshmen. Calculus was to follow in the second volume which was never written.* It is a rigorous pre-calculus text. As Cauchy stated in his preface, "As for the methods, I have sought to give them all of the rigor that one insists upon in geometry." Here we find the first use of $\varepsilon - \delta$ definitions and the first time the modern definitions of continuity and convergence appear in a textbook.

Infinite series were considered to be part of pre-calculus at this time. Cauchy goes to considerable pains to establish them on a solid foundation. He realizes that one can only prove statements about an infinite sum of functions by looking at the approximations by finite sums, and then rigorously justifying the passage to the limit. The first theorem that he proves about infinite sums of functions draws on his definitions of continuity and convergence. He considers a convergent series of continuous functions,

$$S(x) = f_1(x) + f_2(x) + f_3(x) + \cdots ,$$

and defines $S_n(x)$ to be the sum of the first n functions, $R_n(x)$ to be the remainder:

$$S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x),$$
$$R_n(x) = S(x) - S_n(x) = f_{n+1}(x) + f_{n+2}(x) + \cdots .$$

From his definition of convergence, he knows that $R_n(x)$ can be made arbitrarily small by taking n sufficiently large. Though he will use the phrase "infinitely small," it is clear from other contexts that he means "arbitrarily small."

Cauchy had previously proven that any finite sum of continuous functions is continuous, so $S_n(x)$ is continuous. He then goes on to state:

> Let us consider the changes in these three functions when we increase x by an infinitely small value α. For all possible values of n, the change in $S_n(x)$ will be

*The second volume was never written because his students protested so vociferously against this book. What Cauchy was doing was good mathematics, but it was totally inappropriate for his audience.

infinitely small; the change in $R_n(x)$ will be as insignificant as the size of $R_n(x)$ when n is made very large. It follows that the change in the function $S(x)$ can only be an infinitely small quantity. From this remark, we immediately deduce the following proposition:

Theorem I—*When the terms of a series are functions of a single variable x and are continuous with respect to this variable in the neighborhood of a particular value where the series converges, the sum $S(x)$ of the series is also, in the neighborhood of this particular value, a continuous function of x.*[**8**, p. 120]

Cauchy has just proven that any infinite series of continuous functions is continuous.

Five years later, in a footnote to a paper on the binomial series, Niels Henrik Abel wrote about this result, "It appears to me that this theorem suffers exceptions" [**1**]. Indeed, everyone by then knew it did. Fourier series provide the classic examples of infinite sums of continuous functions that are not themselves continuous.

In my classes, this flawed theorem is an opportunity for students to dissect Cauchy's proof in search of the error. Some of them find it. Most of them do not. That is fine. After Abel's observation, it took the mathematical community more than twenty years to formally identify the missing assumption in Cauchy's proof. What is important is that my students are now actively engaged with the ideas behind the theorem. The context has been prepared for the concept of uniform convergence. Students are now able to recognize its importance and usefulness.

Student reactions to the story of Cauchy and the concept of uniform convergence have been instructive. The first time I sprang this on a class, it was greeted with astonishment. How could a great mathematician have been wrong? One student reacted by stating that if he had known earlier that mathematicians could make such fundamental mistakes, then he never would have chosen mathematics as his major. The common reaction was the question, "Then how do we know what is true in mathematics?" There lies the opportunity to begin training mathematicians.

I do need to acknowledge that most historians of mathematics believe that when Cauchy spoke of convergence of a series of functions, he was assuming uniform convergence. But he did not make this clear. This is precisely Lakatos's point. There is an important distinction between *convergence* and *uniform convergence* that was not recognized until the mathematical community began to wrestle with the counter-examples to Cauchy's theorem. We cannot expect our students to appreciate this distinction until they too have had to confront the problems that arise from ignoring it.

Hawkins' challenge

In the spring of 1997, I taught the second semester of our Real Analysis course. All twelve of the students had used my text for their first semester. Where do we go from there?

I decided to stick to the historical theme that they had enjoyed and which had worked so well, but now to aim for measure theory and the Lebesgue integral. A good historical guide to the Lebesgue integral exists in Thomas Hawkins' *Lebesgue's Theory of Integration: Its Origins and Development* [**11**]. This is challenging reading for those who know measure theory. It was not at all clear that it would work as a textbook. I supplemented it with Bartle's *The Elements of Integration and Lebesgue Measure* [**2**].

The class met twice a week for thirteen weeks for an hour and a half each time. We began the semester with Bartle's chapters on Lebesgue measure, spending one class each on outer measure (chapters 11 and 12), measurable sets (chapter 13), Borel sets (chapter 14 and part of 15), approximation and additivity (remainder of chapter 15 and

16), and nonmeasurable sets (chapter 17). The structure of these classes was a mixture of short lecture by me and student presentation of solutions to pre-assigned problems that were based on the material for that day. The pace was intentionally fast. I wanted students to be familiar with the terminology and principal results of measure theory. Depth of understanding would come later.

For the next nine weeks, the class was immersed in Hawkins' book, and I stayed away from the front of the classroom. Since I had twelve students, I had chosen twenty-four nineteenth century mathematicians and twenty-four issues for individuals to explain to the rest of the class. Most days, two students presented reports that included a brief biography of the mathematician and a discussion of the issue that had been assigned.

They gave me their reports three days before the class so that I could duplicate them and get them to the other students in advance. In most cases, students had come to my office at least once before the report was due to ask questions and get clarification of the key ideas. In class, the students presenting reports were grilled by me and the other students. Those presenting were also the local experts for questions about the problems that had been assigned for that day.

The students found Hawkins' book extremely frustrating. Since they came into this course without knowing the theorems that late nineteenth century mathematicians had struggled to discover, they began by taking all of Hawkins's assertions at face value. But Hawkins is recounting history, not describing mathematical facts.

Repeatedly, after seeing a result that seemed to make sense and seeing an argument that looked reasonable, they would then encounter—two or three pages later—a counter-example. And sometimes it was not really a counter-example, it had just seemed like one to the mathematicians of the time. Hawkins' proofs are sketchy, and it takes close reading to distinguish among a proof, a piece of a proof, an outline of a proof, an extended example, and a justification of a "fact" that is later revealed to be wrong. My students constantly felt wrong-footed. More importantly, they realized that they were not alone.

They began to consider themselves part of that community of nineteenth century mathematicians who were wrestling with the concept of the integral. They saw the necessity of Cantor's development of set theory and sympathized with his contemporaries who could make little sense of what he was doing. They found the error in Lipschitz's [**13**] assertion that any nowhere dense set must have a finite number of limit points. They admired Torsten Brodén's construction [**6**], built on ideas of Dini, Köpcke, and Cantor, of a function with a bounded, non(Riemann)-integrable derivative. They agonized over Harnack's inability to recognize the limitations of outer content.

More importantly, my students began to ask their own questions: If a set is closed and has measure zero, can it have positive outer content? As the semester progressed, more and more time was spent on discussion of questions that the students themselves raised.

The issue of outer content versus outer measure emerged as one of the dominant themes of the course. The former covers the set in a finite number of intervals. The latter allows a countable number of intervals. For twenty years, mathematicians focused on outer content rather than measure as they attempted to extend the Riemann integral to functions unbounded around infinitely many points.

My students knew why. Riemann's definition of integrability is expressed in terms of a finite number of subintervals. There is no reason to suspect that a countable number of subintervals is even appropriate. Moreover, in 1885 Axel Harnack "proved" that outer measure is internally inconsistent, that it is dependent on the choice of cover [**10**]. Of course, he did not really prove this, but his published argument did convince many of his contemporaries.

My students came to realize how much more difficult it is to use outer content and that it creates problems for any attempt to extend the integral. After the course was over, one student told me that he had wanted to go back to the nineteenth century and shake some sense into those guys; tell them to stop focusing on finite covers and start looking at countable covers.

In the last weeks of the semester, as we started chapter 5 on Lebesgue integration, I returned to Bartle, using his chapters 2–5 to supplement Hawkins' treatment. The last three reports were on Lebesgue, Baire, and Fubini, and these students were responsible for presenting the proofs of the principal theorems of Lebesgue integration. Borel's measure theory and Lebesgue's integral emerged as shafts of light at the end of a very dark passage. My students embraced Bartle's *Elements of Integration* as they could not have at the beginning of the class. Here at last everything was clear and unambiguous, and they could appreciate the struggle that had gone into establishing this clarity.

For the last week, I turned my students loose on Bartle's "Return to the Riemann Integral" [**3**], a description of and argument for the Henstock integral. They had little trouble digesting this paper and quickly entered into a debate on the pros and cons of this approach to integration.

This was a tough course, but all twelve students stuck with it. The final exam included writing an essay on the problems associated with the Fundamental Theorem of Calculus and how they were overcome. One of the students began, "Before elaborating on the problems with the Fundamental Theorem and their dissolution, I would only like to note that, perhaps for the first time, I got a very real sense of how plausible the Kuhn-Lakatos etc. picture of scientific and mathematical progress is." Another ended his exam with the comment, "Thanks for the class! It has, indeed, been delightfully bewildering, and a real treat. . . . I'll miss it."

Postscript

The lesson is not that everyone should use Hawkins' *Lebesgue's Theory of Integration*, or even my own book that grew out of this experience, *A Radical Approach to Lebesgue's Theory of Integration* [**5**], as the textbook for an advanced real analysis course. Rather, it is that if we want to facilitate real and effective learning in our classes, then we must help our students appreciate that confusion and uncertainty are part of the process of building mathematical knowledge. This includes forcing them to experience some of the confusion and uncertainty that has gone into the creation of mathematics.

At the same time, we must be careful not to overwhelm them. It is our responsibility as teachers to know when to leave them to struggle and when and how to support them as they work through to a personal and meaningful understanding. A textbook or a curriculum is merely a starting point, a framework enabling a good teacher to begin formulating the experiences and challenges that will work in this particular place, at this particular time, with this particular group of students.

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Summary. For far too many students, Real Analysis is a dreaded course that proceeds from unmotivated definitions to formal and impenetrable theorems with little sense of why the course unfolds as it does. This article describes the experience of a radically different approach that drew on the history of mathematics to confront students with the uncertainties and confusion that real mathematicians encounter as they explore new mathematical territory. The argument is made that such an historical approach enables students to appreciate the precise definitions that have evolved and the power of the theorems that are built upon them.

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This has become the touchstone for all of his teaching and his textbooks in number theory, combinatorics, calculus, and analysis. He uses history to understand the problems and questions that motivated the creation of the mathematics to be studied. He then strives to build the text around solving these problems and answering these questions.

Henry Segerman: Visualizing Topology

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Q: *Much of your art is sculptural, using 3D printing as a medium. How did you first get into 3D printing? What was the motivation?*

HS: I was in grad school studying geometry, topology, and three-dimensional maths when I first learned about a virtual world—a video game called Second Life. In the Second Life metaverse, you can build things. The building tools are built into the client, the way that you interact with the world. You can even script things to be interactive in various ways. Nobody at the time was making strange knots or other topological things that I was thinking about in my work, so I started making them. This was my first experience with a 3D design program. I built up a catalog of interesting 3D shapes hidden away inside of this weird game.

Figure 1 As the joke goes, a topologist can't tell the difference between a coffee mug and a donut. This sequence of deformations were generated using conformal Willmore flow. Joint work with K. Crane.

Q: *Could you use the code you wrote for Second Life to help you 3D print the objects?*

HS: You couldn't at the time directly export the designs, but I knew the things that I had made in Second Life looked good, and it wouldn't be so hard to make them again in a different setting. For example, I made an approximation to the Hilbert curve, but in three dimensions. It's a three dimensional space-filling curve. Instead of the usual Hilbert curve that's a squiggle filling up space in a square in the plane, you're looking

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at a curve in a 3D lattice. I made this Hilbert curve approximation in Second Life as a sculpture, and it was a rigid, virtual object. Later, I made it in real life using a software called Rhinoceros and then sent it to Shapeways—an online 3D printing service—to print it. When it came, it was all springy! I wasn't expecting that! It was a real surprise.

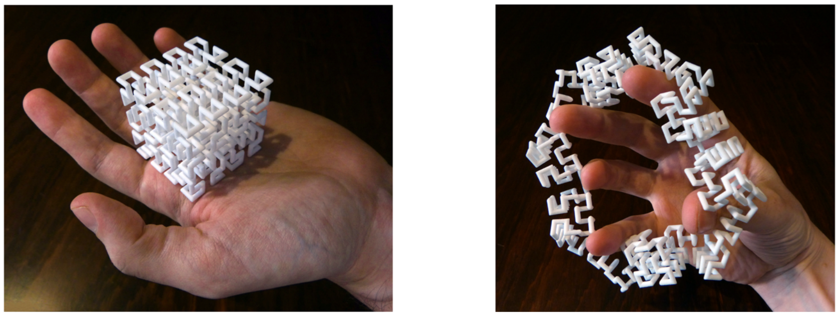

Figure 2 A 3D printed Hilbert cube.

Q: *Do you view the 3D objects you make as art or as tools to explain mathematical ideas?*

HS: A bit of both. A lot of what I make is illustrating some mathematical phenomenon, like the Hilbert curve. Although some of the things I've been doing more recently—like making gears—are more like interesting engineering challenges: can you make things that move in a certain way? For instance, there's this community of people who make crazy Rubik's Cube variants. Oskar van Deventer has this amazing YouTube channel where every week, he shows some new bizarre variant on the Rubik's Cube. And a lot of what he does is about the mechanics of them, but he's also interested in strange mechanisms that are not yet part of a puzzle or some other useful mechanism. But is it art? I resisted the term for a while, but people kept calling me a mathematical artist. There's not much crossover between the mathematical art world and the fine art world. I guess people like Erik and Martin Demaine have crossed that bridge. If people want to call me an artist, that's fine. But if people say, "Oh, you're not an artist!" that's also fine with me. I just do things that are interesting to me.

Q: *Has the process of making 3D printed objects helped you understand math better?*

HS: Inevitably, yes. When you're coding something up or designing it on a computer, you have to really know exactly what's going on. So, you get more comfortable with the concepts when you're telling your computer how to visualize them. But having gone through the design process, when you are holding the 3D print in your hands, you often don't learn more than you did in the design process, though there are occasional surprises.

Q: *Do you have a favorite piece that you've created?*

HS: I guess I think the most effective piece is the stereographic projection grid. A stereographic projection is a map from the sphere to the plane. Imagine a sphere sitting on top of a plane. Now, draw a line from the north pole of the sphere down through the sphere to the plane. This line hits the sphere at a point p and it hits the plane at a point q, and the map sends p to q. You can describe the function relatively easily, but it's not so obvious from that description how it changes shapes or distorts things. The sculpture of the stereographic projection grid mimics the projection with a flashlight; the flashlight does precisely what the map does. Light hits the sphere somewhere,

Figure 3 A relatively common sight in graphic design is a planar arrangement of three gears in contact. However, since neighboring gears must rotate in opposite directions, none of the gears can move. This sculpture gives a non-planar, and non-frozen, arrangement of three linked gears. Joint work with S. Schleimer.

and it hits the plane somewhere. This phenomenon has been made visual with computer renders, but people are rightly suspicious that renders can be faked. With the 3D printed model, you can't cheat. It's a bit like a magic trick, particularly if you show somebody the print beforehand and then you move the light into the right place. Then it's a surprise that this thing with this weird curvy grid is actually hiding the flat regular grid. Just like with a magic trick, people want to know what's going on. But unlike magicians, mathematicians want to say, "Yes! We can tell you what's going on! You just have to learn what stereographic projection is!"

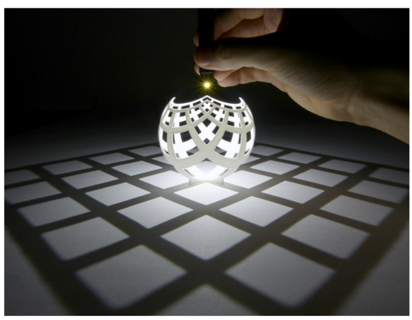

Figure 4 The stereographic projection of a spherical design onto a planar grid.

Q: *Can you tell me a little bit about your book?*

HS: Sure! It's intended to be an accessible maths book. There's not much in the way of equations and formulas and so on. It illustrates some really beautiful mathematics, like stereographic projection, visually. Most of the figures in the book are photographs of real 3D-printed things, and there's a website that goes along with the book, 3dprintmath.com, which has a virtual version of the 3D-printed objects that you can rotate around. You can also download the files for your own 3D printer or order the 3D-printed object online. The idea is that the figures in the book are 3-dimensional. For certain kinds of things you want to explain, it helps to see them in 3D. We cover several topics, like tilings and curvature, knots and topology, and so on. Ideally, you'd be sitting with the book and you'd have all the figures on your shelf. You could just grab, say, Figure 5.6, and you'd have it in your hand as you were reading the book.

Q: *More recently, you've collaborated with Elisabetta Matsumoto on virtual reality projects. These projects basically involve every single aspect of STEAM (science, tech-*

Figure 5 The gears on this version of Buckminster Fuller's expanding and contracting "jitterbug" mechanism help to keep the parts aligned, making the movement smoother. Joint work with S. Matsumoto.

nology, engineering, art, and mathematics). How did the collaboration came about, and how did you learn what you needed to learn in order to start working in virtual reality?

HS: I first got into this through Vi Hart. She had a research group that was interested in virtual reality called eleVR. I knew Vi through the Bridges mathematical conferences. I went to visit her team over one winter break and started learning how to do it then. I later met Sabetta, and she's interested in a lot of these same visualization ideas and 3D printing, and so we just naturally started collaborating as well.

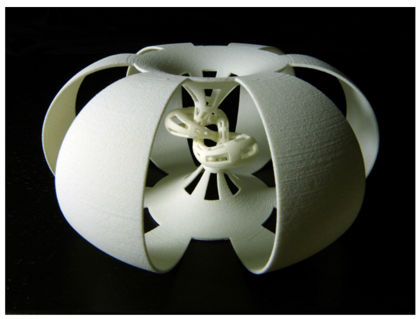

Figure 6 This sculpture illustrates the complement of the figure-eight-knot in the 3-sphere. The knot is parameterized in the 3-sphere, and then stereographically projected to 3-space. Joint work with F. Guéritaud and S. Schleimer.

Q: *Currently, what do the projects that you've done in VR help visualize?*

HS: The first one is hypernom ("hyper" as in the hypersphere, and "nom" as in "nom, nom, nom"), which is a game where you try and eat the cells of a 4-dimensional polytope. Making the game involved visualizing the 3-sphere—the sphere that lives in 4-dimensional space. A polyhedron in three dimensions can be projected radially onto the ordinary 2-sphere in three dimensions, giving a nice tiling of the sphere. The same thing can be done in one dimension up. In the game, you can see the 3-sphere as if you're inside of it. You see light rays travel along great circles, which are geodesics in the sphere. In the game, you have to try and eat all of the polytopes, and it's controlled by the orientation of your headset.

The orientation of your headset is given by a 3×3 matrix with orthogonal rows. This space is the same topologically as $\mathbb{R}P^3$, real projective space, which is double-covered by the 3-sphere. What that means is that when you move your head around, it's moving you through the 3-sphere! You navigate the space by rotating your head around. In order to eat all of the cells, you have to get your head into every orientation twice because of this double cover. It's a fun, silly game that can help you learn about topological phenomena. More recently, we've simulated 3-dimensional hyperbolic space. Sabetta and I also made a version which was 2-dimensional hyperbolic and 1-dimensional Euclidean. Philosophically, you want to use the medium—whatever the medium is—to do things that you wouldn't be able to do in any other medium. The physicality of the 3D-print and the flashlight seems really powerful for illustrating stereographic projection. What VR is good for is something that there's no physical structure you could build that would illustrate it. So, if the space is really curved around you and you're somehow moving through it—short of getting together a bunch of tame black holes—it's not going to happen in real life.

Figure 7 This is possibly the first sculpture with the quaternion group as its symmetry group. The monkey was designed in a 3D cube, viewed as one of the eight cells of a hypercube. The quaternion group moves the monkey to the other seven cells. Radial projection moves the monkeys onto the 3-sphere—the unit sphere in 4D space—and then stereographic projection moves the monkeys to 3D space. Joint work with W. Segerman.

Q: *What advice do you have for people who would like to get into these areas but don't have the technological expertise yet to do 3D printing or virtual reality?*

HS: For 3D printing, there is a lot of software out there. For instance, Rhinoceros is good professional software that people like architects use a lot, but it costs quite a bit of money. There are much easier things you can play around with. Laura Taalman is the real expert on this! But I think that she'd recommend Tinkercad, a website that lets you play around with primitives and put them together. You can then export something that can be 3D printed. That's free and accessible on the Web. There's another software called OpenSCAD, which has you write simple code to generate geometry. For virtual reality, I was lucky because Vi Hart's team already had things up and running, and I could look at their code and mess around with it. So, find someone who is already doing what you want to do and ask if you can learn from them!

Yet Another Proof Without Words of the Pythagorean Theorem

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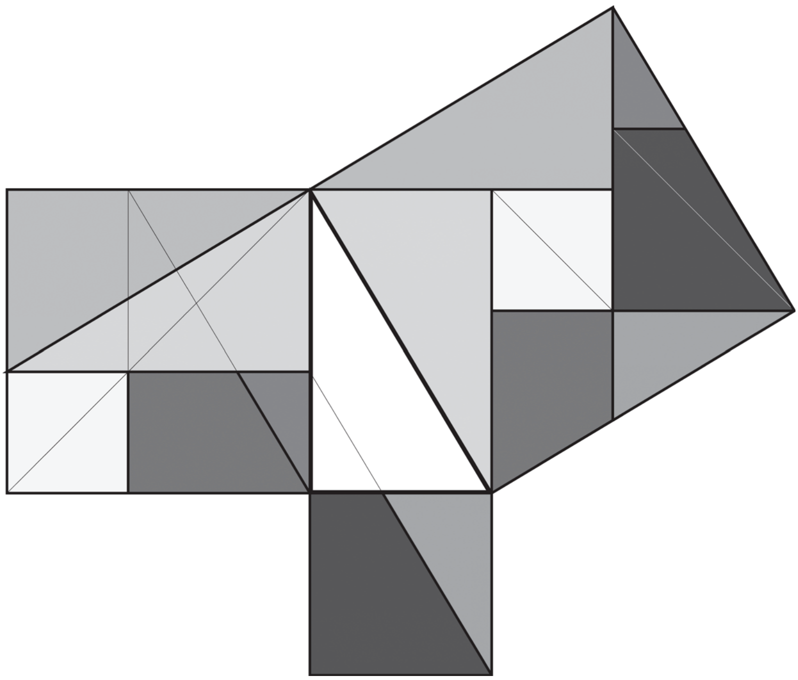

Summary. We present a visual, dissection proof for the Pythagorean theorem.

Domenico Cantone (MR Author ID: 44895) is a professor of computer science at the University of Catania, Italy. He received his Ph.D. degree from New York University in 1987, under the supervision of Prof. Jacob T. Schwartz. His main scientific interests include the following: computable set theory, automated deduction in various mathematical theories, description logic, string matching and algorithm engineering, and, more recently, rational choice theory from a logical point of view.

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MSC: 51-01

An Identity from Viète

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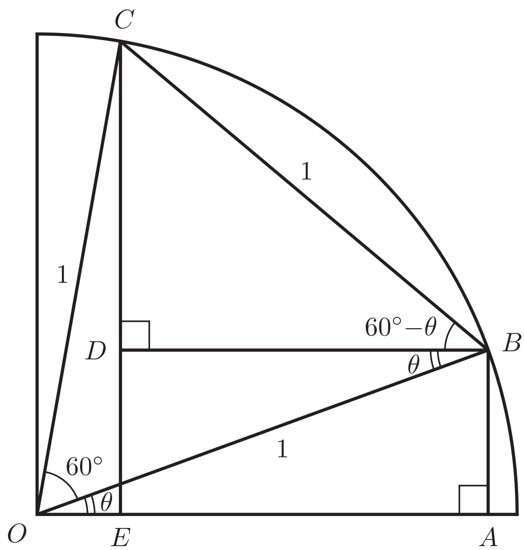

$$AB = CE - CD \Longrightarrow \sin\theta = \sin(60^\circ + \theta) - \sin(60^\circ - \theta)$$

$$OA = OE + BD \Longrightarrow \cos\theta = \cos(60^\circ + \theta) + \cos(60^\circ - \theta)$$

The first identity appears without a proof in *Universalium inspectionum*, the second part of Viète's first published mathematical work *Canon Mathematicus*. He used this identity in the computation of his trigonometric tables. He remarked that once we know the sines up to 30°, the rest can be obtained by simple addition and subtraction [**1**, p. 10]. However, this does not appear to be sufficient.

REFERENCE

[1] Viète, F. (1579). *Canon mathematicus seu ad triangula cum adpendicibus, Universalium inspectionum ad canonem mathematicum liber singularis*. Lvtetiæ: Apud Ioannem Mettayer. https://books.google.com/books?id=BJJq8IHp-zQC

Summary. We provide a visual proof for a sine identity used by Viète in the computation of his sine table. The figure also proves the counterpart for the cosine function.

REX H. WU (MR Author ID: 1293646) is interested in the visualization of mathematical identities and concepts. He wishes to thank Professors Henry Ricardo and Lawrence D'Antonio, the anonymous reviewers and the editor for their suggestions. His salute goes to all the healthcare providers fighting the COVID-19 pandemic.

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MSC: 97G60

Half Row Sums in Pascal's Triangle

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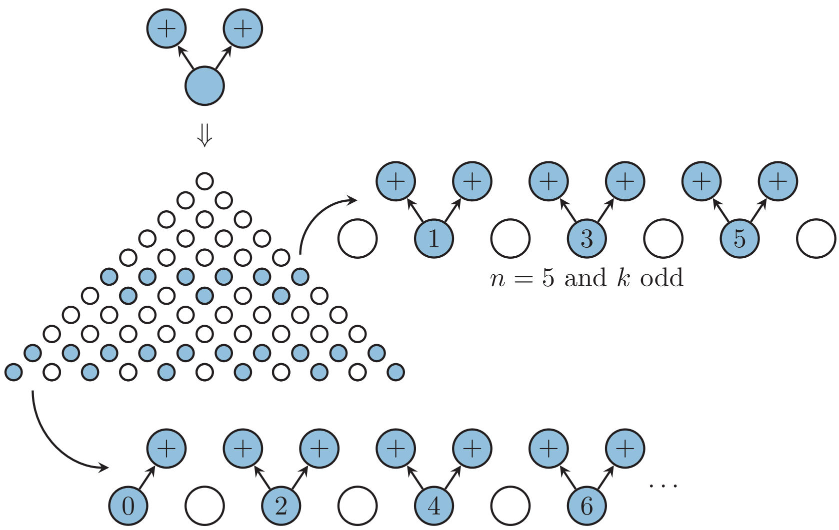
Theorem. *For any integer $n \geq 0$,*

$$\sum_{k \text{ even}} \binom{n+1}{k} = \sum_{k \text{ odd}} \binom{n+1}{k} = 2^n.$$

Proof.



Summary. We demonstrate visually that the sum of every other term in the $(n+1)$st row of Pascal's triangle is equal to the sum of all the terms in the previous row.

ANGEL PLAZA (MR Author ID: 350023) received his masters degree from Universidad Complutense de Madrid in 1984 and his Ph.D. from Universidad de Las Palmas de Gran Canaria in 1993, where he is full professor in applied mathematics. He is interested in mesh generation and refinement, combinatorics and visualization support in teaching and learning mathematics.

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MSC: Primary 05A19
Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/umma.

PROBLEMS

Proposals

To be considered for publication, solutions should be received by March 1, 2021.

2101. *Proposed by Michael Goldenberg, The Ingenuity Project, Baltimore Polytechnic Institute, Baltimore, MD and Mark Kaplan, Towson University, Towson, MD.*

Recall that the Steiner inellipse of a triangle is the unique ellipse that is tangent to each side of the triangle at the midpoints of those sides. Consider the Steiner inellipse E_S of $\triangle ABC$ and another ellipse, E_A, passing through the centroid G of $\triangle ABC$ and tangent to $\overleftrightarrow{AB}$ at B and to $\overleftrightarrow{AC}$ at C. If E_S and E_A meet at M and N, let $\angle MAN = \alpha$. Construct ellipses E_B and E_C, introduce their points of intersection with E_S, and define angles β and γ in an analogous way. Prove that

$$\frac{\cot\alpha + \cot\beta + \cot\gamma}{\cot A + \cot B + \cot C} = \frac{11}{3\sqrt{5}}.$$

2102. *Proposed by Donald Jay Moore, Wichita, KS.*

Let $\alpha = \pi/7$, $\beta = 2\pi/7$, and $\gamma = 4\pi/7$. Prove the following trigonometric identities.

$$\frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\beta}{\cos^2\gamma} + \frac{\cos^2\gamma}{\cos^2\alpha} = 10,$$

$$\frac{\sin^2\alpha}{\sin^2\beta} + \frac{\sin^2\beta}{\sin^2\gamma} + \frac{\sin^2\gamma}{\sin^2\alpha} = 6,$$

$$\frac{\tan^2\alpha}{\tan^2\beta} + \frac{\tan^2\beta}{\tan^2\gamma} + \frac{\tan^2\gamma}{\tan^2\alpha} = 83.$$

Math. Mag. **93** (2020) 309–318. doi:10.1080/0025570X.2020.1801039.

We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this MAGAZINE.

Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at http://mathematicsmagazine.submittable.com. *More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by LaTeX source. General inquiries to the editors should be sent to* mathmagproblems@maa.org.

2103. *Proposed by Péter Kórus, University of Szeged, Szeged, Hungary.*

In a soccer game there are three possible outcomes: a win for the home team (denoted 1), a draw (denoted X), or a win for the visiting team (denoted 2). If there are n games, betting slips are printed for all 3^n possible outcomes. For four games, what is the minimum number of slips you must purchase to guarantee that at least three of the outcomes are correct on at least one of your slips?

2104. *Proposed by the Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.*

It is well known that no vector space can be written as the union of two proper subspaces. For which m does there exist a vector space V that can be written as a union of m proper subspaces with this collection of subspaces being minimal in the sense that no union of a proper subcollection is equal to V?

2105. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.*

Let $f : [0, 1] \to \mathbb{R}$ be a function that is k times differentiable on [0, 1], with the kth derivative integrable on [0, 1] and (left) continuous at 1. For integers $i \geq 1$ and $j \geq 0$ let

$$\sigma_j^{(i)} = \sum_{j_1+j_2+\cdots+j_i=j} 1^{j_1} 2^{j_2} \cdots i^{j_i},$$

where the sum is extended over all i-tuples $(j_1, \ldots, j_i)$ of nonnegative integers that sum to j. Thus, for example, $\sigma_0^{(i)} = 1$, and $\sigma_1^{(i)} = 1 + 2 + \cdots + i = i(i + 1)/2$ for all $i \geq 1$. Also, for $0 \leq j \leq k$ let

$$a_j = \sigma_j^{(1)} f(1) + \sigma_{j-1}^{(2)} f'(1) + \cdots + \sigma_1^{(j)} f^{(j-1)}(1) + \sigma_0^{(j+1)} f^{(j)}(1).$$

Prove that

$$\int_0^1 x^n f(x) dx = \frac{a_0}{n} - \frac{a_1}{n^2} + \cdots + (-1)^k \frac{a_k}{n^{k+1}} + o\left(\frac{1}{n^{k+1}}\right),$$

for $n \to \infty$. As usual, we denote by $f^{(s)}$ the sth derivative of f (with $f^{(0)} = f$), and by $o(x_n)$ a sequence (y_n) with the property that $\lim_{n\to\infty} y_n/x_n = 0$.

Quickies

1103. *Proposed by Elias Lampakis, Kiparissia, Greece.*

Let a, b, and c be the side lengths of a triangle, r its inradius, and R its circumradius. Show that

$$a^2b^2 + b^2c^2 + c^2a^2 \geq 108R^2r^2.$$

1104. *Proposed by George Stoica, Saint John, NB, Canada.*

Prove that, for every positive real number a, there exists a sequence $k_1, k_2, \ldots$ of positive integers such that $\{a \cdot 1^{k_1} \cdot 2^{k_2} \cdot \cdots \cdot n^{k_n}\} < 1/n$ for all $n \geq 1$. (Here $\{x\}$ denotes the fractional part of x.)

Solutions

Tribonacci and dual Tribonacci sequences **October 2019**

2076. *Proposed by Michael Goldenberg, The Ingenuity Project, Baltimore Polytechnic Institute, Baltimore, MD and Mark Kaplan, Towson University, Towson, MD.*

Given real numbers C_0, C_1, and C_2, one defines a *general Tribonacci (GT) sequence* $\{C_n\}$ recursively by the relation $C_{n+3} = C_{n+2} + C_{n+1} + C_n$ for all $n \geq 0$. Such GT-sequence $\{C_n\}$ is *nonsingular* if

$$\Delta = \begin{vmatrix} C_0 & C_1 & C_2 \\ C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \end{vmatrix} \neq 0.$$

A *dual Tribonacci (DT) sequence* $\{D_n\}$ is one that satisfies the dual recurrence $D_{n+3} + D_{n+2} + D_{n+1} = D_n$ for $n \geq 0$. Show that for any nonsingular GT-sequence $\{C_n\}$ with C_0, C_1, C_2 positive there exists a DT-sequence $\{D_n\}$ such that, for all $n \geq 0$,

$$\arctan\left(\frac{\sqrt{D_n}}{C_n}\right) = \arctan\left(\frac{\sqrt{D_n}}{C_{n+1}}\right) + \arctan\left(\frac{\sqrt{D_n}}{C_{n+2}}\right) + \arctan\left(\frac{\sqrt{D_n}}{C_{n+3}}\right).$$

Composite of solutions by Robert Calcaterra, University of Wisconsin-Platteville, Platteville, WI and Albert Stadler, Herrliberg, Switzerland.

Observe that the sequence where $D_n = 0$ for all n clearly satisfies the conditions of the problem. We will therefore endeavor to find a non-trivial solution to the problem.

The addition formula for the tangent function can be extended to show that

$$\tan(x + y + z) = \frac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - \tan x \tan y - \tan x \tan z - \tan y \tan z}.$$

Therefore, the sequence D_n must satisfy

$$\frac{\sqrt{D_n}}{C_n} = \frac{\frac{\sqrt{D_n}}{C_{n+1}} + \frac{\sqrt{D_n}}{C_{n+2}} + \frac{\sqrt{D_n}}{C_{n+3}} - \frac{D_n^{3/2}}{C_{n+1}C_{n+2}C_{n+3}}}{1 - \frac{D_n}{C_{n+1}C_{n+2}} - \frac{D_n}{C_{n+1}C_{n+3}} - \frac{D_n}{C_{n+2}C_{n+3}}}. \tag{1}$$

If $D_n \neq 0$, we can divide both sides of the equation by $\sqrt{D_n}$ and solve for D_n to get

$$D_n = \frac{C_nC_{n+1}C_{n+2} + C_nC_{n+1}C_{n+3} + C_nC_{n+2}C_{n+3} - C_{n+1}C_{n+2}C_{n+3}}{C_n - C_{n+1} - C_{n+2} - C_{n+3}}.$$

Since C_n is a GT-sequence,

$$C_n - C_{n+1} - C_{n+2} - C_{n+3} = -2\left(C_{n+1} + C_{n+2}\right)$$

and

$$\begin{aligned} &C_nC_{n+2}C_{n+3} + C_nC_{n+1}C_{n+3} + C_nC_{n+1}C_{n+2} - C_{n+1}C_{n+2}C_{n+3} \\ &= C_nC_{n+2}C_{n+3} + C_nC_{n+1}C_{n+3} + C_{n+1}C_{n+2}(C_n - C_{n+3}) \\ &= C_nC_{n+2}C_{n+3} + C_nC_{n+1}C_{n+3} - C_{n+1}C_{n+2}(C_{n+1} + C_{n+2}) \\ &= (C_nC_{n+3} - C_{n+1}C_{n+2})(C_{n+1} + C_{n+2}). \end{aligned}$$

Therefore,

$$D_n = \frac{C_{n+1}C_{n+2} - C_nC_{n+3}}{2}.$$

Because all of the steps above are reversible, this does satisfy equation (1).

Since D_n may be negative, we recall that for real x

$$\arctan ix = \begin{cases} i \operatorname{arctanh}\ x & |x| < 1, \\ \operatorname{sign}(x)(\pi/2 + i \operatorname{arctanh}(1/|x|)) & |x| > 1. \end{cases}$$

Note that equation (1) may not translate into the desired statement about arctangents. For example, if $C_0 = 8$, $C_1 = 3$, $C_2 = 2$, then $C_3 = 13$ and $D_0 = -49$. We have

$$\arctan\left(\frac{\sqrt{D_0}}{C_0}\right) = \frac{\ln(15)}{2} i,$$

but

$$\arctan\left(\frac{\sqrt{D_0}}{C_1}\right) + \arctan\left(\frac{\sqrt{D_0}}{C_2}\right) + \arctan\left(\frac{\sqrt{D_0}}{C_3}\right) = \pi + \frac{\ln(15)}{2} i.$$

We will return to this issue shortly.

By the theory of linear recurrences, there are complex numbers a_1, a_2, a_3 such that

$$C_n = a_1 r_1^n + a_2 r_2^n + a_3 r_3^n.$$

where r_1 is the real root and r_2, r_3 the conjugate complex roots of the equation $x^3 - x^2 - x - 1 = 0$. Note that by Vieta's formulas, $r_1 r_2 r_3 = 1$.

Now

$$\begin{aligned} 2D_n &= C_{n+1}C_{n+2} - C_n C_{n+3} \\ &= -a_1 a_2 (r_1 - r_2)^2 (r_1 + r_2) r_3^{-n} - a_2 a_3 (r_2 - r_3)^2 (r_2 + r_3) r_1^{-n} \\ &\quad - a_3 a_1 (r_3 - r_1)^2 (r_3 + r_1) r_2^{-n}. \end{aligned}$$

Since D_n is a linear combination of powers of $1/r_1$, $1/r_2$, and $1/r_3$, it satisfies a linear recurrence whose characteristic polynomial has these values as roots. Therefore, the characteristic polynomial is $x^3 + x^2 + x - 1$ and D_n satisfies $D_{n+3} + D_{n+2} + D_{n+1} = D_n$ as desired.

The condition $\Delta \neq 0$ is equivalent to

$$a_1 a_2 a_3 (r_1 - r_2)^2 (r_2 - r_3)^2 (r_3 - r_1)^2 \neq 0,$$

which means that $a_1, a_2, a_3 \neq 0$ and D_n is non-trivial. Finally, $|\sqrt{D_n}|$ grows like $1/\sqrt{|r_2|}^n$, while C_n grows like r_1^n, so $\sqrt{D_n}/C_n$ grows like $1/(r_1\sqrt{|r_2|})^n$. Since

$$r_1\sqrt{|r_2|} \approx 1.579, \lim_{n\to\infty} \sqrt{D_n}/C_n = 0$$

and for sufficiently large n we will be able to translate equation (1) into one involving arctangents.

Also solved by Elias Lampakis (Greece), Daniel Văcaru (Romania), and the proposers. There were two incomplete or incorrect solutions.

A triangle inequality **October 2019**

2077. *Proposed by Li Zhou, Polk State College, Winter Haven, FL.*

Prove that in any triangle with side lengths a, b, c, inradius r, and circumradius R, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{r}{R}>\frac{5}{3}.$$

Solution by Celia Schacht (graduate student), North Carolina State University, Raleigh, NC.

We first observe that three real numbers a, b, and c are the side lengths of a triangle if and only if there exist three positive real numbers x, y, and z, such that

$$a=x+y,\quad b=y+z,\quad \text{and}\quad c=x+z.$$

Without loss of generality, we may assume that $x \le y \le z$.

If s denotes the semiperimeter of the triangle and A its area, then in terms of x, y, and z, we have

$$s=x+y+z \quad \text{and} \quad A=\sqrt{xyz(x+y+z)}.$$

It is well known that

$$r=\frac{A}{s} \quad \text{and} \quad R=\frac{abc}{4A}.$$

Rewritten in term of x, y, and z, the original inequality becomes

$$\frac{x+y}{x+y+2z}+\frac{x+z}{x+2y+z}+\frac{y+z}{2x+y+z}+\frac{4xyz}{(x+y)(y+z)(x+z)}>\frac{5}{3}.$$

To prove this we will show that both

$$\frac{x+y}{x+y+2z}+\frac{x+z}{x+2y+z}>\frac{2}{3} \tag{1}$$

and

$$\frac{y+z}{2x+y+z}+\frac{4xyz}{(x+y)(y+z)(x+z)}\ge 1 \tag{2}$$

hold.

Now (1) is equivalent to

$$3(x+y)(x+2y+z)+3(x+z)(x+y+2z)>2(x+y+2z)(x+2y+z)$$

which can be rewritten as

$$2x^2+3xy+3xz+(y-z)^2>0.$$

Since x, y, and z are positive, this holds.

Note that (2) is equivalent to

$$(y+z)^2(x+z)(x+y)+4xyz(2x+y+z)\ge(2x+y+z)(y+z)(x+z)(x+y),$$

which (since $x>0$) is equivalent to

$$2xyz+y^2z+yz^2\ge x^2y+xy^2+x^2z+xz^2.$$

This can be easily seen to be true, by noticing that, since $x \le y \le z$, we have

$$xyz \ge x^2z, xyz \ge xy^2, yz^2 \ge xz^2, y^2z \ge x^2y.$$

The original inequality is sharp, as one can see by putting $a = c$, $b = 2a - \epsilon$, and letting $\epsilon \to 0$.

Also solved by Michel Bataille (France), Robert Calcaterra, Habib Far, Subhankar Gayen (India), Michael Goldenberg & Mark Kaplan, Walther Janous (Austria), Omran Kouba (Syria), Elias Lampakis (Greece), Kee-Wai Lau (Hong Kong), Volkhard Schindler (Germany), Albert Stadler (Switzerland), Daniel Văcaru (Romania), Michael Vowe (Switzerland), John Zacharias and the proposer.

A nilpotent commutator — October 2019

2078. *Proposed by Florin Stanescu, Şerban Cioculescu School, Găeşti, Romania.*

Let A, B be $n \times n$ complex matrices such that $A^2 + B^2 = 2AB$. Prove that $(AB - BA)^m = \mathbf{0}$ for some $m \le \lceil \frac{n}{2} \rceil$.

Solution by Koopa Tak Lun Koo, Chinese STEAM Academy, Hong Kong.
Let $X = A - B$ and rewrite the given condition as $X^2 = AB - BA$. We first show that X is nilpotent. Rewriting the given condition as $X^2 = XB - BX$, and multiplying both sides by X^{k-1} on the right gives

$$X^{k+1} = XBX^{k-1} - BX^k.$$

Thus

$$\mathrm{tr}(X^{k+1}) = \mathrm{tr}(XBX^{k-1}) - \mathrm{tr}(BX^k) = \mathrm{tr}(BX^{k-1} \cdot X) - \mathrm{tr}(BX^k) = 0$$

for all $k \ge 1$.

If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of X, then

$$\mathrm{tr}(X^\ell) = \sum_{i=1}^{n} \lambda_i^\ell = 0$$

for all $\ell \ge 2$. This forces $\lambda_i = 0$ for all i, so the characteristic polynomial for X is λ^n. By the Cayley–Hamilton theorem, $X^n = \mathbf{0}$.

Let $m = \lceil n/2 \rceil$. Then $(AB - BA)^m = X^{2m} = \mathbf{0}$ since $2m \ge n$ and we are done.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert Calcaterra, Eugene Herman, John C. Kieffer, Julio Cesar Mohnsam (Brazil), Northwestern University Problem Solving Group, Daniel Văcaru (Romania), and the proposer. There was one incomplete or incorrect solution.

An improper integral that almost never converges — October 2019

2079. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Given real numbers a, b with $b > 0$, prove that the integral

$$J(a, b) := \int_0^\infty \left[2 + (x + a)\ln\left(\frac{x}{x + b}\right)\right] dx$$

converges if and only if $a = 1$ and $b = 2$, and find the value $J(1, 2)$.

Solution by Eugene Herman, Grinnell, IA.
Let $F(x)$ denote an antiderivative of the integrand. Integrate by parts using the antiderivatives $x(x+2a)/2$ and $(x+2a-b)(x+b)/2$ of $x+a$:

$$\begin{aligned}
F(x) &= 2x + \int (x+a)\ln x\,dx - \int (x+a)\ln(x+b)\,dx \\
&= 2x + \frac{1}{2}x(x+2a)\ln x - \int \frac{1}{2}(x+2a)\,dx \\
&\quad - \frac{1}{2}(x+2a-b)(x+b)\ln(x+b) + \int \frac{1}{2}(x+2a-b)\,dx \\
&= 2x + \frac{1}{2}x(x+2a)\ln x - \frac{1}{2}(x+2a-b)(x+b)\left(\ln x - \ln\left(\frac{x}{x+b}\right)\right) - \frac{bx}{2}
\end{aligned}$$

Note that $\lim_{x\to 0^+} F(x) = -(2a-b)b(\ln b)/2$. So convergence of the integral depends entirely on convergence of $\lim_{x\to\infty} F(x)$. We use

$$\ln(1-t) = -t - \frac{1}{2}t^2 + O(t^3) \quad \text{as } t \to 0^+$$

and so

$$\ln\left(\frac{x}{x+b}\right) = \ln\left(1 - \frac{b}{x+b}\right) = -\frac{b}{x+b} - \frac{1}{2}\left(\frac{b}{x+b}\right)^2 + O(1/x^3) \quad \text{as } x \to \infty$$

Hence

$$\begin{aligned}
F(x) &= 2x - \frac{bx}{2} + \frac{b^2-2ab}{2}\ln x + \frac{x+2a-b}{2}(x+b)\left(-\frac{b}{x+b} - \frac{1}{2}\left(\frac{b}{x+b}\right)^2\right) + O\left(\frac{1}{x}\right) \\
&= 2x - \frac{bx}{2} + \frac{b^2-2ab}{2}\ln x - \frac{(x+2a-b)b}{2} - \frac{(x+2a-b)b^2}{4(x+b)} + O\left(\frac{1}{x}\right)
\end{aligned}$$

In order that $\lim_{x\to\infty} F(x)$ converge, the coefficients of x and $\ln x$ must be zero. Thus, $2 - b/2 - b/2 = 0$ and $b(b-2a)/2 = 0$, and so $b = 2$ and $a = 1$. Therefore, since $\lim_{x\to 0^+} F(x) = 0$,

$$J(1,2) = \lim_{x\to\infty} -\frac{4x}{4(x+2)} = -1$$

Also solved by Ulrich Abel (Germany), Michel Bataille (France), Paul Bracken, Brian Bradie, Robert Calcaterra, Hongwei Chen, Paul Deiermann, Shuyang Gao, Finbarr Holland (Ireland), Walther Janous (Austria), Elias Lampakis (Greece), Missouri State University Problem Solving Group, Angel Plaza (Spain), Arthur Rosenthal, Albert Stadler (Switzerland), Daniel Văcaru (Romania), and the proposers. There were three incomplete or incorrect solution.

Edge colorings with no monochromatic triangles **October 2019**

2080. *Proposed by the UTSA Problem Solving Club, University of Texas at San Antonio, San Antonio, TX.*

For $n \geq 3$, let W_n be the wheel graph consisting of an n-cycle all whose vertices are joined to an additional distinct vertex.

(*i*) How many colorings of the $2n$ edges of W_n using $k \geq 2$ colors result in no monochromatic triangles?

(*ii*) Regard two colorings of W_n as equivalent if there is a graph automorphism of W_n that maps the first coloring to the second. If $k \geq 2$ and $p > 3$ is prime, count all non-equivalent colorings of W_p using k colors.

Solution by Rob Pratt, Apex, NC.

(i) We apply the principle of inclusion and exclusion (PIE). For $n = 3$, there are six edges and four triangles. Ignoring monochromaticity, there are k^6 edge colorings. For each monochromatic triangle, there are k ways to color the triangle and k^{6-3} ways to color the remaining edges. For each pair of monochromatic triangles (which must share an edge), there are k ways to color the triangles and k ways to color the remaining edge. For three (equivalently, four) monochromatic triangles, there are k ways to color the triangles and no remaining edges. So PIE yields

$$k^6 - \binom{4}{1}k \cdot k^{6-3} + \binom{4}{2}k \cdot k - \binom{4}{3}k + \binom{4}{4}k = k^6 - 4k^4 + 6k^2 - 3k$$

edge colorings with no monochromatic triangles.

For $n > 3$, there are n triangles. For $t < n$ monochromatic triangles with s shared edges, there are k^{t-s} ways to color the triangles and $k^{2n-3t+s}$ ways to color the remaining edges. If all n triangles are monochromatic, there are k colorings. Now PIE yields

$$\begin{aligned}
&\sum_{t=0}^{n-1}(-1)^t\binom{n}{t}k^{t-s} \cdot k^{2n-3t+s} + (-1)^n k \\
&= \sum_{t=0}^{n}(-1)^t\binom{n}{t}(k^2)^{n-t} - (-1)^n\binom{n}{n}(k^2)^{n-n} + (-1)^n k \\
&= (k^2-1)^n + (-1)^n(k-1).
\end{aligned}$$

(ii) We apply the Cauchy–Frobenius–Burnside theorem, which states that the number of equivalence classes of a finite set X under the action of a finite group G is

$$\frac{1}{|G|}\sum_{g\in G} F_g,$$

where $F_g = |\{x \in X : g \cdot x = x\}|$ is the cardinality of the set of fixed points of $g \in G$.

The automorphism group of the wheel graph W_p is the dihedral group of order $2p$. The identity element fixes all k^{2p} edge colorings. Because $p > 3$ is prime, each nontrivial rotation fixes only the colorings for which both the p spokes and the sides of the outer p-cycle are monochromatic, yielding k^2 colorings. Each reflection fixes only the colorings for which the $(2p-2)/2 = p-1$ edge pairs across the reflection are monochromatic, and the two self-reflective edges can take any color. Hence the number of non-equivalent colorings is

$$\frac{1}{2p}\left(k^{2p} + (p-1)k^2 + p \cdot k^2 \cdot k^{p-1}\right) = \frac{k^{2p} + (p-1)k^2 + p \cdot k^{p+1}}{2p}.$$

Editor's Note. The proposers' intent was that part (ii) also required triangles to be non-monochromatic, but this was not explicitly stated. Here is a solution to the intended problem by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela.

(ii) Denote the central vertex by v_0 and the vertices of the n-cycle by $v_1, \ldots, v_n$. Let r_i denote the spoke $v_0 v_i$ and call the edges $v_1 v_2, \ldots, v_n v_1$ sides.

The identity leaves fixed all the colorings, namely $(k^2-1)^p - k + 1$ of them by part (i).

Since p is prime, any of the $p-1$ rotations other than the identity is a generator of the group of rotations, hence for a coloring to be fixed, all the spokes must be of the same color and also all the sides must have the same color (different from the color of the spokes). Thus there are $k(k-1)$ such colorings.

Let $m = (p+1)/2$. There are $\binom{m-1}{j}$ ways to choose exactly j pairs of consecutive spokes from $r_1, \ldots, r_m$ with the same color. This gives $m-j$ clusters of adjacent spokes each cluster having the same color. There are $k(k-1)^{m-j-1}$ ways to color these clusters. This gives a total of $\binom{m-1}{j}k(k-1)^{m-1-j}$ ways to color $r_1, \ldots, r_m$ with exactly j pairs of consecutive spokes having the same color. Now consider a reflection, say the one through v_1. Each of the spoke colorings can be extended to a coloring of W_p without monochromatic triangles and invariant under this reflection in $(k-1)^{j+1}k^{m-1-j}$ ways (the exponent of $(k-1)$ is $j+1$ to account for the color of the side $v_m v_{m+1}$). Hence the number of colorings invariant under this reflection is

$$\sum_{j=0}^{m-1} \binom{m-1}{j} k^{m-j}(k-1)^m = k(k-1)^m (k+1)^{m-1} = k(k-1)^{\frac{p+1}{2}}(k+1)^{\frac{p-1}{2}}.$$

Finally the number of non-equivalent colorings is

$$= \frac{1}{2p}\left((k^2-1)^p - k + 1 + (p-1)k(k-1) + pk(k-1)^{\frac{p+1}{2}}(k+1)^{\frac{p-1}{2}}\right)$$
$$= \frac{1}{2}k(k-1)\left(1 + (k^2-1)^{\frac{p-1}{2}}\right) + \frac{1}{2p}(k^2-1)\left((k^2-1)^{p-1} - 1\right).$$

We note that the case $p = 3$ may be handled with the same techniques. The group of automorphisms is isomorphic to S_4 and the number of non-equivalent colorings is

$$\frac{1}{24}k^3(k-1)(k^3+k^2+6k-6).$$

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), and the proposers.

Answers

Solutions to the Quickies from page 310.

A1103. It is well known that

$$r = \frac{2A}{a+b+c} \quad \text{and} \quad R = \frac{abc}{4A},$$

where A denotes the area of the triangle. Therefore the inequality we wish to prove is equivalent to

$$a^2b^2 + b^2c^2 + c^2a^2 \geq 27\frac{a^2b^2c^2}{(a+b+c)^2}$$

or equivalently

$$\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)(a+b+c)^2 \geq 27.$$

Hölder's inequality is

$$\left(\sum_{i=1}^{k} x_i^p\right)^{1/p}\left(\sum_{i=1}^{k} y_i^q\right)^{1/q} \geq \sum_{i=1}^{k} x_i y_i,$$

where x_i, $y_i \geq 0$ and $1/p+1/q=1$. Let $k=3$, $p=3$, $q=3/2$, and

$$x_1=\frac{1}{a^{2/3}},\ x_2=\frac{1}{b^{2/3}},\ x_3=\frac{1}{c^{2/3}},\ y_1=a^{2/3},\ y_2=b^{2/3},\ y_3=c^{2/3}.$$

Then Hölder's inequality becomes

$$\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^{1/3}(a+b+c)^{2/3} \geq 3$$

or equivalently

$$\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)(a+b+c)^2 \geq 27,$$

which is what we wanted to show. Note that the case when $a=b=c$ shows that this inequality is sharp.

A1104. Let us denote by a_n the fractional part of $a \cdot 1^{k_1} \cdot 2^{k_2} \cdot \cdots \cdot n^{k_n}$, $n \geq 1$.

Put $k_1=1$ and assume that we have already defined $k_2, \ldots, k_n$ such that $a_1<1$, $a_2<1/2, \ldots, a_n<1/n$. Keeping in mind that

$$\frac{1}{n}=\sum_{i=1}^{\infty}\frac{1}{(n+1)^i},$$

we either have that

$$a_n<\frac{1}{n+1}, \tag{1}$$

or there exists a unique positive integer $p \geq 1$ such that

$$\frac{1}{n+1}+\cdots+\frac{1}{(n+1)^p} \leq a_n < \frac{1}{n+1}+\cdots+\frac{1}{(n+1)^{p+1}}. \tag{2}$$

In the first case, put $k_{n+1}=0$; in the second case put $k_{n+1}=p$.

In both cases, a_{n+1}, which is the fractional part of the product $a_n \cdot (n+1)^{k_{n+1}}$, is less than $\frac{1}{n+1}$, by either (1) or (2), and the induction process is complete.

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

D'Agostino, Susan, *How to Free Your Inner Mathematician: Notes on Mathematics and Life*, Oxford University Press, 2020; xi+350 pp, $32.95. ISBN 978-0-19-884359-7.

In an era of excellent books about mathematics for a popular audience, this book stands out, for its spirit, imagination, and liveliness. The author "dropped out of mathematics" after failing a high school calculus exam, studied anthropology and film, and farmed. But she "pined to learn more mathematics," and did, attaining a Ph.D. The titles of the book's 46 chapters combine aphorisms with mathematical content: "Find balance, as in coding theory"; "Work efficiently, like bacteriophages with icosahedral symmetry"; "Consider all options, as the shortest path between two points is not always straight." And each chapter ends with a brief elaboration on a lesson for pursuits in life and in mathematics. The chapters are short, full of neat hand-draw figures, and end with a single exercise (rarely two), with solutions provided. D'Agostino's theme is that "Everyone has significant, untapped mathematical potential. . . . [C]uriosity, desire, and persistence matter more in mathematics than innate talent." She urges readers to "Choose to observe life through a mathematical lens," and that is the perspective that this book richly offers. Unfortunately, the book is marred by mischaracterizing Hilbert's Third Problem as asking whether two tetrahedra with equal bases and altitudes can have different volumes (they can't). A quibble with the copy editor: "Königsberg," "Möbius," and "Dalí" should have gotten their diacritical marks; "Gödel" came out right.

Daley, Beth, An inmate's love for math leads to new discoveries, theconversation.com/an-inmates-love-for-math-leads-to-new-discoveries-130123.

Havens, Christopher, Podcast: Inmate Haven: What pi means to me, www.doc.wa.gov/news/docs/2017-0317-podcast-inmate-haven.pdf.

________, Stefano Barbero, Umberto Cerruti, and Nadir Murru, Linear fractional transformations and nonlinear leaping convergents of some continued fractions, *Research in Number Theory* 6, 11 (2020), link.springer.com/article/10.1007/s40993-020-0187-5.

A high school dropout, a convicted murderer in solitary confinement in a U.S. prison, discovered in himself a passion for mathematics. Over a period of only one year—at 10 hours a day of study—he taught himself some advanced mathematics. With encouragement from a mathematician, he solved a problem involving continued fractions and is the first author of the resulting publication. He tells his story in the podcast cited.

Lample, Guillaume, and François Charton, Using neural networks to solve advanced mathematics equations, ai.facebook.com/blog/using-neural-networks-to-solve-advanced-mathematics-equations/.

________, Deep learning for symbolic mathematics, arxiv.org/pdf/1912.01412.pdf.

A new approach to solving integral and differential equations is based on turning an algebraic expression into a tree, as if its elements were parts of speech, and then applying research from machine translation of languages. The output from the authors' trained neural network is not

Math. Mag. **93** (2020) 319–320. doi:10.1080/0025570X.2020.1792241. © Mathematical Association of America

guaranteed to be correct, and the network cannot explain the "reasoning" involved. But the method achieved more successes, in much shorter times, than Mathematica and Matlab.

Kucharski, Adam, *The Rules of Contagion: Why Things Spread—And Why They Stop*, Basic Books, 2020; 343 pp, $30. ISBN 978-1541674318.

This book, written before the emergence of Covid-19 and of the Black Lives Matter movement, details the principles underlying any form of contagious outbreak, whether spread of a disease, a scientific concept, an attitude, crime, violence, a tweet, fake news, a movement, or a computer virus. The book is rich in details about instances of each of these. Author Kucharski examines all these manifestations of contagion and reflects on the underlying commonalities and the differing dynamics. Not much mathematics is involved, beyond exponential growth, a formula for the reproduction number R, and a discussion of small-world networks.

Niss, Mogens, and Werner Blum, *The Learning and Teaching of Mathematical Modelling*, Routledge, 2020; viii+199 pp, $155, $44.95(P), $42.70(ebook), $8.46(ebook rental). ISBN 978-1-138-73067-0, 978-1-138-73070-0, 978-315-18931-4.

This is an important resource for teachers of mathematical modeling. It offers a general framework for the fundamental ideas involved in both mathematical modeling and the enterprise of teaching it, and it sums up and explains the relevant empirical research. It contains concrete illustrations of classroom modeling activities and highlights several case studies of implementation of mathematical modeling in mathematics curricula. Among points that struck me as important are the emphasis on the role of "pre-mathematisation" (selecting elements of the situation relevant to the questions asked), the challenges presented by pre-existing expectations of students and teachers, and the difficulty of transfer of learning from one domain to another. A surprise was to learn that Gustave Flaubert offered in 1841 a problem very much like the recently-notorious "How old is the shepherd?" problem: "There are 125 sheep and 5 dogs in a flock. How old is the shepherd?" Three-quarters of eighth-graders "made sense" of the problem by reasoning such as "$125 + 5 = 130 \ldots$ too big $\ldots$ $125 - 5 = 120 \ldots$ still too big $\ldots$ $125/5 = 25 \ldots$ that works" (www.youtube.com/watch?v=kibaFBgaPx4).

Romano, Alessandro, Chiara Sotis, Goran Dominioni, and Sebastián Guidi, The public do not understand logarithmic graphs used to portray COVID-19, blogs.lse.ac.uk/covid19/2020/05/19/the-public-doesnt-understand-logarithmic-graphs-often-used-to-portray-covid-19/.

The assertion of the title of this article will not be a surprise to instructors of mathematics! The authors describe an experiment with 2,000 U.S. residents, with half provided information on progression of Covid-19 deaths presented on a logarithmic scale and half presented with a "good old linear scale." Only half as many logarithmic respondents could answer correctly a simple question about the graph; and that group made worse predictions for deaths in the week following the data, though both groups asserted the same level of confidence in their answers. Worse, the scale they saw affected respondents' attitudes: the logarithmic group was less worried about the Covid-19 crisis. However, the authors go beyond the data to claim that "unlike the people who saw the graph on a logarithmic scale, the people exposed to a linear scale graph can form their preferences based on information that they can understand better."

Brown, Ezra, and Richard K. Guy, *The Unity of Combinatorics*, MAA Press, 2020; xiv+353 pp, $65. ISBN 978-1-4704-5279-7.

The authors' purpose is to illustrate that combinatorics is not just "a bag of tricks in many areas … with little or no connection between them." Brown and Guy address this mission via examples, whose investigations chain naturally from one concept to another, but they do not identify any underlying unifying principles. The first chapter pursues Langford sequences to Skolem sequences to Beatty sequences to Conway worms in aperiodic tilings; the second chapter takes off from Beatty sequences to a constellation of combinatorial games. Further chapters explore a cornucopia of combinatorial concepts. The book began with a 1994 lecture by Guy (who died recently at age 103), and here is expanded by subsequent publications (11 of the 19 chapters), with the remaining chapters being Guy–Brown collaborations for this book.